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THE FEBRUARY MEETING IN NEW YORK

The two hundred fifty-third regular meeting of the Society was held at Columbia University, on Saturday, February 26, 1927, extending through the usual morning and afternoon sessions. The attendance included the following seventy-two members:

W. L. Ayres, W. M. Bond, Carrie, L. D. Cummings, Dadourian, Dehn, Doermann, Douglas, Eisenhart, Ettlinger, Fiske, Fite, D. A. Flanders, Fort, Gehman, Gill, Glenn, Gronwall, C. C. Grove, Guggenbühl, Haskins, Hedlund, Himwich, Joffe, R. A. Johnson, Kasner, Kellogg, Kholodovsky, Kline, Koopman, Kuhn, Langman, Lefschetz, Littauer, Luby, MacColl, MacNeish, Michael, T. W. Moore, Mullins, F. H. Murray, K. E. O'Brien, Pederson, Pfeiffer, Pierpont, Pooler, Post, R. G. Putnam, Raudenbush, Reddick, Reid, R. G. D. Richardson, D. E. Richmond, Ritt, Seely, Skolnik, Smail, M. H. Stone, Takasu, E. M. Thomas, T. Y. Thomas, Tracey, Vallarta, Veblen, Evelyn Walker, Wedderburn, Weida, Weisner, Whittemore, Wiener, W. A. Wilson, J. W. Young.

There was no meeting of the Council or of the Trustees of the Society.

Professor Wiener presided at the morning session, and Vice-President Kellogg in the afternoon.

At the request of the Program Committee, Professor D. J. Struik, of the Deft Technical School, now visiting professor at the Massachusetts Institute of Technology, delivered an address, at the beginning of the afternoon session, entitled *The geometry of linear displacements*; this paper will be published in full in an early issue of this Bulletin.

Titles and abstracts of the other papers read at this meeting follow below. The papers of Ayers (first paper), Cramlet, Franklin, Garver, Hollcroft, Rosebrugh, Silverman and Tamarkin, Stone, Thomas, G. T. Whyburn, W. M. Whyburn, Widder, Wiener (first three papers), and Wilder were read by title. Mr. Gourin was introduced by Professor Ritt, and Professor Rosebrugh by the Secretary.

1. Professor H. J. Ettlinger: *On sequences of horizontal (step) functions.*

If E is a subset of points of a cube S in m dimensions, and S is subdivided into subcells by means of m sets of n hyperplanes parallel to the fundamental coordinate planes, then a horizontal function of index $k=n^m$ is a function defined on E which is constant at all points of the subset E contained in a given subcell of S. An infinite sequence of horizontal functions is a set of horizontal functions defined for $n=1, 2, 3, \cdots$. If a function defined on E is the limit almost everywhere of an infinite sequence of horizontal functions, it is called a function of class M. It is the object of the present paper to make a systematic study of the properties of functions of class M.

2. Dr. T. H. Gronwall: *Longitudinal vibrations of a liquid contained in a tube with elastic walls.*

An experimental method for determining the velocity of sound in a liquid consists in closing the lower end of a vertical metal tube by a membrane which is excited by an alternating current of known frequency, and filling the tube with liquid up to the point where resonance is obtained. The characteristic equation determining the frequencies of the vibrations of the liquid is set up, in the form of an infinite determinant, the elements of which contain products of Bessel functions, and, from this, approximate formulas of simple form are derived, which are directly applicable to the experimental data.

3. Dr. T. H. Gronwall: *On the existence and properties of the solutions of a certain differential equation of the second order.*

Under certain very general assumptions on $f(x, y)$ and $g(x, y)$, it is shown that the differential equation $(d/dx)[g(x, y)dy/dx]=f(x, y)$ with any one of the four boundary conditions (A) $y(x_0)=y_1$, $y(x_1)=y_1$; (B) $g(x_0, y(x_0))y'(x_0)=\text{const.}$, $y(x_1)=y_1$; (C) $y(x_1)=y_1$, $y(x)\to 0$ as $x\to\infty$; (D) $g(x_0, y(x_0))y'(x_0)=\text{const.}$, $y(x)\to 0$ as $x\to\infty$, has a solution which is unique. It is shown that any solution has at most one extreme, and a very general comparison theorem for the solutions of two equations of this type is established. A special case of the above equation with boundary conditions (D) occurs in the Debye-Hückel theory of solutions of strong electrolytes.

4. Dr. T. H. Gronwall: *On the converse of Euler's theorem for homogeneous functions.*

A homogeneous function $\phi(x_1, x_2, \cdots, x_n)$ of degree k satisfies Euler's partial differential equation (where m is any integer)

$$\sum x_{\nu_1} x_{\nu_2} \cdots x_{\nu_m} \phi_{\nu_1,\nu_2,\cdots,\nu_m} = k(k-1)\cdots(k-m+1)\phi,$$

where the subscripts on ϕ denote partial derivatives, and the summation

extends over all subscripts of sum m. It is shown that the general solution of this equation is

$$\phi = \sum \phi^{(\rho)}(x_1, \cdots, x_m) + \log x_1 \cdot \sum{}' \psi^{(\rho)}(x_1, \cdots, x_m),$$

where $\phi^{(\rho)}$ and $\psi^{(\rho)}$ are arbitrary homogeneous functions of degree ρ, and the first sum extends over all the roots ρ of $\rho(\rho-1)\cdots(\rho-m+1) = k(k-1)\cdots(k-m+1)$, while the second sum extends the over all double roots (if any; roots of higher multiplicity do not occur).

5. Dr. Jesse Douglas (National Research Fellow): *Reduction to integral equations of the problem of Plateau for the case of two contours.*

Let $x_i = {}^1f_i(t)$, $x_i = {}^2f_i(t)$, $i = 1, 2, \cdots, n$, represent the given two contours Γ_1, Γ_2, supposed contained in a euclidean space of n dimensions. The $2n$ functions f are presumed to be periodic of common least positive period Ω. The problem of Plateau is formulated so as to require a doubly-connected portion M of a minimal surface bounded by Γ_1, Γ_2, and at the same time a conformal map of M on a circular ring, outer radius 1, inner radius, originally unknown, q. When Γ_1, Γ_2 are the boundaries of a doubly-connected portion of a plane, this reduces to the Riemann mapping problem. This paper reduces the Plateau problem to a system of two integral equations, for two functions $\phi_1(t)$, $\phi_2(t)$ of the points of Γ_1, Γ_2 respectively, together with a third equation for the determination of $p = \log(1/q)$.

6. Dr. Jesse Douglas: *Extremals and transversality of the general calculus of variations problem of the first order in space.*

Every calculus of variations problem,

$$(1) \quad \int F(x, y, z, y', z')dx = \text{minimum},$$

defines two geometric objects: first, its extremals, a family F of ∞^4 curves in space; second, its transversality T, which is essentially a certain correspondence between lineal elements λ and surface elements σ such that corresponding elements have the same base point. The present paper establishes a simple criterion for determining of a given family F of ∞^4 curves, together with a given transversality T, whether or not they can be identified respectively with the extremals and transversality of a problem of the form (1). The criterion is that the ∞^2 curves of F which meet any one surface (union) Σ transversality must always admit ∞^1 transversal surfaces. The proof is rather long, and will be published in the April, 1927, number of the Transactions of this Society.

7. Mr. Eli Gourin: *On the essentiality of arbitrary functions.*

Consider an analytic function $F(p, q, p', q')$, where p, q are arbitrary functions of x and p', q' their respective derivatives. The paper gives a set of conditions necessary and sufficient for the existence of a relation $F(p, q, p', q') = S(\alpha, \alpha')$, where α is a function of p and q and α' is $d\alpha/dx$.

8. Mr. H. W. Raudenbush: *Note on Hilbert's thirteenth problem.*

It is proved that f defined by the equation

$$f^7+xf^3+yf^2+zf+1=0$$

cannot have any one of the following forms: $F(\alpha(x, y), \beta(y, z))$, $F(\alpha(x, y), \beta(x, z))$, or $F(\alpha(x, z), \beta(y, z))$, where α, β, and $F(\alpha, \beta)$ are analytic functions.

9. Professor O. E. Glenn: *A theory of integers in relation to the iteration of algebraic functions.*

The content and methods of this paper are presented in the form of original algorithms on the number system. The object is to develop a theory of indices in which Δ of the fundamental relation $\Delta^r \equiv \Delta \pmod{n}$ instead of being an integer, is a functional operation upon an integer, and Δ^r is its rth iteration. The index r is determined with considerable generality as the serial number of a term in a sequence of resultants. The resultants are modular and appertain to the theory due to Stieltjes. Applications are noted, to problems relating to two-dimensional residue systems.

10. Mr. T. W. Moore: *On the invariant combinants of two binary quintics.* Preliminary report.

In a study of the rational space quintic curve the author has been led to the set of combinant forms of two binary quintics. From work of Gordan and Stroh it is known that the complete system of combinants of the two quintics is the same as the simultaneous system of covariant forms of the binary octavic and quartic, which are the first and third transvectants respectively of the two given forms. In turn, this system is identical, except for invariant factors, with that of another binary octavic apolar to the two quintics, and so the number of combinant forms is known. Starting with the three syzygies given by L. Berzolari in vol. 7 of the Rendiconti di Palermo, this paper sets up the complete system of invariant combinants of the two forms of the fifth order, together with the necessary covariant forms required for this purpose.

11. Dr. Louis Weisner: *Invariant theory of plane n-points and n-lines.*

The author investigates the projective invariant theory of complete n-points and n-lines. The work naturally falls into two parts: the consideration of concomitants expressed in terms of the coordinates of the vertices of the n-points and of the sides of the n-lines, and the consideration of concomitants expressed in terms of the coefficients of the degenerate ternary line-curves and point-curves defined by the n-points and n-lines respectively. Finiteness theorems are proved, a symbolic theory is developed, and a theorem somewhat analogous to Hermite's reciprocity law is proved.

12. Professor James Pierpont: *Optics in spaces of constant curvature.* Second paper.

In a paper read at the last Annual Meeting of the Society, the author showed that in space of constant non-vanishing curvature, central optical collineation gives only an image congruent with the object. Thus this elegant method is not available in elliptic or hyperbolic spaces. However, instead of rays of light one may use waves of light, adopting the methods laid down by C. S. Hastings (*On certain new methods and results in optics*, Memoirs of the National Academy vol. 6 (1893), p. 37). In general it is found that the results are similar to those of euclidean optics. As an example, the familiar formula for a convex lens, $(1/r)+(1/r_1)=((1-n)/n)((1/l)+(1/l_1))$, $n<1$, becomes for hyperbolic spaces, $\text{ctnh}\,(\rho_1/R)+\text{ctnh}(\rho/R)=((1-n)/n)(\text{ctnh}\,(\lambda/R)+\text{ctnh}\,(\lambda_1/R))$, and a similar formula for elliptic space.

13. Dr. M. H. Stone: *A characteristic property of certain sets of trigonometric functions.*

In this paper the nature of a closed normal orthogonal set of functions whose derivatives constitute an orthogonal set is discussed. It is shown that when certain auxiliary conditions are satisfied by the functions of the set they are necessarily trigonometric and satisfy a differential system of the second order.

14. Dr. M. H. Stone: *The normal frequency function and general frequency functions.*

The problem considered is that of approximating to an arbitrary function by a linear combination of functions of the form $e^{-\lambda(x-\mu)^2}$, where λ and μ are constants, $\lambda>0$. Three theorems concerning the possibility of such representation are obtained with the aid of the theorem of Riesz-Fischer.

15. Professor Norbert Wiener: *On the closure of sets of trigonometric functions.*

The author discusses the closure of the set of functions $\cos \lambda_n x$.

16. Professor Norbert Wiener: *A new definition of almost periodic functions.*

The author gives a new necessary and sufficient condition for a function to be almost periodic.

17. Professor Norbert Wiener: *Laplacians and linear functionals.*

The author develops a new theory of functions of limited total variation in n dimensions, connected with the Laplacian operator. He applies this to the theory of linear functionals and to that of multi-dimensional trigonometric developments.

18. Mr. G. T. Whyburn: *Concerning the complementary domains of continua.*

The author shows that (1) if the complementary domains of a boundel continuum M have the property that for any number $\epsilon>0$ at most a finite number of them are of diameter $>\epsilon$, and P is any point of M not belonging to the boundary of any complementary domain of M, then (a) for every number $d>0$, M contains a subcontinuum of diameter $<d$ which separates P from all those points of the plane whose distance from P is $>d$, and (b) M is regular at P; (2) the complementary domains of a bounded continuum M have the above mentioned property if and only if for every point P of M and every number $d>0$ there exists a continuum N of diameter $<d$ which separates P from all those points of the plane whose distances from P are $>d$ and which is a subset of $M+$ a finite number of the complementary domains of M; and (3) if the complementary domains of an indecomposable continuum M have the above mentioned property, then M is the boundary of at least one of its complementary domains.

19. Mr. G. T. Whyburn: *Concerning the sum of the boundaries of the complementary domains of a plane continuous curve.*

In this paper it is shown that if H denotes the point set obtained by adding together the boundaries of all the complementary domains of a continuous curve, M every subcontinuum of which is a continuous curve, then (1), the set $M-H$ contains no continuum, and (2), the set H is connected. It is further shown that if M is any continuous curve which contains no domain, then in order that the point set H obtained by adding together the boundaries of all the complementary domains of M should be connected, it is necessary and sufficient that the set $M-H$ should contain no closed subset which separates the plane.

20. Mr. G. T. Whyburn: *Concerning the open subsets of a continuous curve.*

A boundary point P of a domain D in a space E will be said to be *regularly accessible* from D provided it is true that for every positive number ϵ, a $\delta_\epsilon>0$ exists, such that every point X of D, whose distance from P is $<\delta_\epsilon$, may be joined to P by an arc XP of diameter $<\epsilon$, and such that $XP-P$ lies in D. In this paper it is shown that (1) if E is the euclidean plane, the above definition is equivalent to accessibility from all sides as introduced by Schoenflies; (2) if E is any plane continuous curve, every boundary point of a domain D is regularly accessible from D if and only if D has property S (R. L. Moore, Fundamenta Mathematicae, vol. 3, p. 232); (3) if E is a plane continuous curve, every subcontinuum of which is a continuous curve, every boundary point of any domain D is regularly accessible from D; (4) a connected open subset of a plane continuous curve is cyclically connected if and only if it has no cut point; (5) if M and N are continuous curves, $N\subset M$, then every maximal connected subset of $M-N$ has property S.

21. Professor W. M. Whyburn: *Existence and oscillation theorems for second-order differential systems whose boundary conditions contain definite integrals.*

In this paper, the pair of differential equations $dy/dx = K(x, \lambda)z$, $dz/dx = G(x, \lambda)y$, together with boundary conditions, one of which contains a definite integral (Lebesgue), is treated. The functions K and G are assumed continuous in λ for each x on the interval of definition, and summable in x for each λ on the λ interval. For every x and every λ on the intervals of definition, K and G are less than (or equal to) a summable function of x. In the presence of the restrictions that K/G and A/G be non-decreasing functions of x for each fixed λ, where $A(x, \lambda)$ is the coefficient of $y(x, \lambda)$ in the integral condition, the existence of an infinite set of characteristic values of λ is established. Oscillation theorems are given for the solutions of the system, for linear combinations of these solutions, and for indefinite integrals containing these solutions in their integrands. Similar theorems are proved for $z(x, \lambda)$.

22. Professor W. M. Whyburn and Mr. G. T. Whyburn: *On the Porter-Vitali theorem.*

The authors give a new and shorter proof of a theorem due to Arzela that every bounded sequence of functions which are analytic in a closed region R contains a subsequence which converges to an analytic limit function everywhere in R. On the basis of this theorem they obtain a very short proof of the Porter-Vitali theorem that if a uniformly bounded sequence of functions analytic in a region R converges over a set of points having a limit point in R, then the whole sequence converges throughout R to an analytic limit function. A new application of these theorems is given.

23. Professor R. L. Wilder: *A point set which has no true quasi-components, and which becomes connected upon the addition of a single point.*

Sierpinski has given (Fundamenta Mathematicae, vol. 2 (1921), pp. 81–85) an example of a point set which has no true quasi-components, but which, on the addition of a single point, contains a quasi-component. In the present paper the author shows, by an example, that there exists a point set M which has no true quasi-component, and which, on the addition of a single point, becomes connected. Incidentally, the set M furnishes another example of a certain type of point set recently constructed by Mazurkiewicz (Fundamenta Mathematicae, vol. 2, pp. 201–205); i. e., a quasi-connected set M of which every two points are separated in M.

24. Mr. W. L. Ayres: *Note on a theorem concerning continuous curves.*

In a number of recent applications the following theorem has been found useful: If M is a plane continuous curve (bounded or unbounded)

and N is a bounded continuous curve which is a subset of M, then, for any positive number ϵ, $M-N$ contains at most a finite number of maximal connected subsets of diameter greater than ϵ. This is an extension of a result previously presented (*Concerning the arcs and domains of a continuous curve*, this Bulletin, vol. 32 (1926), p. 37).

25. Professor T. R. Hollcroft: *On higher point correspondences.*

When both planes are multiple, only (2, 2) and (2, 3) point correspondences have been classified and studied in detail. In this paper, (3, 3) point correspondences between two planes are completely classified. Fifteen types of general (3, 3) point correspondences and fifteen types of (3, 3) compound involutions are found, and the characteristic features of each type are given. The number of types each of (2, 4), (3, 4), and (4, 4) point correspondences and the general characteristics of these correspondences are also found. As in the case of (1, 4) involutions when multiplicity of one or both planes is greater than four, no new features are introduced.

26. Dr. Raymond Garver: *A theorem concerning Tschirnhaus transformations.*

The following theorem is proved: Every Tschirnhaus transformation of degree $n-1$ which transforms a sufficiently general principal equation (in x) of degree $n>4$ into another principal equation can be expressed as a linear function of ϕ and ψ, where ϕ and ψ are polynomials (in x) with special properties. (By a principal equation we mean one in which the second and third coefficients are zero.) Transformations of this kind are especially convenient to work with, and we thus show that in certain problems we suffer no loss of generality by restricting our attention to them.

27. Dr. Raymond Garver: *Division algebras of order sixteen.*

In this paper a type of division algebra of order sixteen, based on a root of a quartic equation irreducible in the field of rational numbers and having the Galios group (1), (12) (34), (13) (24), (14) (23), is set up, in accordance with the conditions given in Dickson's *New division algebras* (Transactions of this Society, April, 1926). For this case there are six conditions, which restrict the coefficients of the quartic and the constants of multiplication of the algebra. Cases are exhibited where all the conditions are satisfied.

28. Mr. C. M. Cramlet: *The derivation of projective algebraic invariants by tensor algebra.*

In this paper it is proved that any algebraic invariant can be readily written as an inner product of the fundamental tensors, which are the coefficients of the forms, and the determinant tensor

$$\delta^{r_1 \cdots r_n}_{s_1 \cdots s_n}.$$

The methods apply also to non-symmetric tensors. Although invariants written in this manner are explicitly expressed, the transition to the classical symbolic form involves a mere change of notation.

29. Dr. D. V. Widder (National Research Fellow): *The transformations of the special relativity theory.*

In the special relativity theory it is desired to determine those real one-to-one transformations $x_i = x_i(y_1, y_2, y_3, y_4)$ which make

$$dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 = \mu(dy_1^2 + dy_2^2 + dy_3^2 - dy_4^2),$$

where μ is a function of y_1, y_2, y_3, y_4, in general different from zero. In this paper, all possible values of μ are determined, and the corresponding transformations are set up. It is found that the case $\mu =$ constant is the only one that leads to a transformation continuous throughout space. This continuity is a requirement of the relativity theory, so that in this way all other values of μ may be excluded.

30. Dr. J. M. Thomas (National Research Fellow): *On systems of total differential equations.*

It is shown that, in applying the theorem given by O. Veblen and the writer (Annals of Mathematics, (2), vol. 27 (1926), p. 290) to systems of tensor differential equations, the process of differentiation can be chosen as partial or covariant at will. By adopting the latter alternative, the theorems for the existence of homogeneous first integrals in the geometry of paths (O. Veblen and T. Y. Thomas, Transactions of this Society, vol. 25 (1923), pp. 551–608) are readily deduced as special cases.

31. Professor J. W. Young: *On the definition of the Scorza-Dickson algebras.*

The definition of "an algebra A over a field F" given by Dickson in his *Algebras and their Arithmetics*, pp. 9, 10, consists of five groups of postulates I-V, of which groups II and III refer exclusively to multiplication between the elements of A and the marks of F. In the present paper it is shown that for a large part at least, if not for all, of the later theory the postulate groups II and III are unnecessary and all reference to a field F can be avoided. A slight modification of postulate group I (insuring merely that A forms an abelian group with respect to addition) and postulate groups IV and V constitute a sufficient basis from which to derive all the theorems in the first three chapters of Dickson's book. This is accomplished by using the idea of the partitions of the addition group (see abstract of a paper by the author presented to the Society September 9, 1926, this Bulletin, vol. 32, p. 588).

32. Professor Norbert Wiener and Dr. Robert Schmidt: *A general form of Tauberian theorem.*

By using the Fourier integral and the related theory of complex moments, the authors obtain a very general Tauberian theorem, in-

cluding most known Tauberian theorems as special cases, in particular, the Tauberian theorem for Lambert series, from which the prime number theorem may be deduced, as Hardy has shown.

33. Professor Norbert Wiener: *The approximation theorem for almost periodic functions.*

The author gives a new proof of the approximation theorem for almost periodic functions.

34. Mr. W. L. Ayres: *On the separation of points of a continuous curve by arcs and simple closed curves.*

If x and y are distinct points of a plane continuous curve M (bounded or unbounded) and M_{xy} denotes the set of all points P, such that P lies on some simple continuous arc of M whose end points are x and y, then M_{xy} is a continuous curve. With the use of this result, the following two theorems are proved: (1) a necessary and sufficient condition that M contain a point which separates x and y in M is that every two simple continuous arcs of M whose end points are x and y have a point distinct from x and y in common; (2) if M is bounded, a necessary and sufficient condition that M contain a *simple continuous arc* which separates x and y in M is that $M-(x+y)$ contain only one maximal connected subset which has both x and y as limit points; (3) if M is bounded, a necessary and sufficient condition that M contain a simple closed curve J, such that one of the two points x or y is interior to J and the other is exterior, is that both x and y do not lie on the boundary of the same complementary domain of M. In (2), a point is considered to be a special case of an arc.

35. Mr. W. L. Ayres: *Concerning the boundaries of domains of a continuous curve.*

If D is a domain of a plane continuous curve M, B is its M-boundary, and P is a point of $M-(D+B)$, then the M-boundary of the maximal connected subset of $M-(D+B)$ containing P is called the M-boundary of D with respect to P. These theorems are then proved: if N is a continuous curve which is a subset of M, every closed and connected subset of the M-boundary of a complementary M-domain of M is a continuous curve; (2) if β is the M-boundary of D with respect to P, then β is the entire M-boundary of some M-domain which contains D; (3) if every maximal connected subset of B is a continuous curve and either D or the maximal connected subset of $M-(D+B)$ which contains P is bounded, then every maximal connected subset of the M-boundary of D with respect to P is either a point, an arc, or a simple closed curve, and if one maximal connected subset is a simple closed curve J, then J is the entire M-boundary of D with respect to P; (4) if β is the M-boundary of D with respect to P, R is the maximal connected subset of $M-(D+B)$ containing the point P, and Q is any point of the maximal connected subset of $M-(R+\beta)$ which contains D, then β is the M-boudnary of R with respect to Q.

36. Professor Edward Kasner: *Equilong geometry.*

In previous papers, the author has studied various questions of equilong geometry mainly with the object of comparison with the analogous (but in general more difficult) questions of conformal geometry (see Proceedings of the Fifth International Congress, Cambridge, 1912, vol. 2, pp. 81–87, and this Bulletin, vol. 23 (1917), pp. 341–347). In the present paper, he discusses more fully the analogy of isothermal systems of curves in equilong geometry, equilong symmetry (see this Bulletin, vol. 24 (1918), p. 471), and the invariants of irregular analytic curves. Some types have invariants and some have not, in general as in the conformal theory (see Transactions of this Society, vol. 16 (1915), p. 339), but the detailed results are different.

37. Professor Edward Kasner: *Irregular differential invariants.* I: *General theory.*

The ordinary concept of differential element and differential invariant relates to curves as represented by polynomials or power series with integer exponents. The author has extended this theory by allowing irregular elements or curves where the series involve fractional exponents. This theory has been carried out for the conformal group of the plane in his paper in the Transactions of this Society, vol. 16 (1915), pp. 333–349 (see also this Bulletin, vol. 21 (1915), p. 280), and for the equilong group in his first paper read at this meeting of the Society. In the present paper he considers continuous groups in general, and in later papers will discuss the irregular invariants of the infinite groups of contact transformations and of point transformations, as well as the finite projective and affine groups.

38. Professor Philip Franklin: *Analytic functions with assigned values at an infinite number of points.*

The question of determining when the values of a function at an infinite set of points in a finite region determine a function analytic in this region, and if so the function in question, has recently been raised by T. H. Hildebrandt (this Bulletin, vol. 32 (1926), p. 552). This question is answered in the present note.

39. Professors L. L. Silverman and J. D. Tamarkin: *On the generalization of Abel's theorem for certain definitions of summability.*

Abel's classical theorem on power series was first generalized by Frobenius, for the case when the series is summable by arithmetic means. Subsequently, Abel's theorem has been extended by various authors to other definitions of summability: Cesàro's and Hölder's of all orders, Borel's and Euler's. The purpose of the present paper is to discuss the extension of Abel's theorem to the general class of analytically regular transformations (Hurwitz and Silverman, Transactions of this Society,

vol. 18 (1917), pp. 1–20); the result obtained is that among these transformations only those equivalent to Hölder's admit of the generalization of Abel's theorem. The general regular transformations permutable with M, as given by Hausdorff, are also discussed, and, finally, Abel's theorem is extended to the definition of summability of Nörlund.

40. Professor T. R. Rosebrugh: *Quantic determinants.*

In this discussion, any number of linear substitutions are given, independent of each other as to coefficients and variables. The substitution is formed which gives all corresponding new monomials linearly in terms of all the possible old monomials whose factors are taken from all substitutions, and from each to a total degree assigned to that substitution. The algebraic composition of this determinant is found without assumptions by examination of the effect of certain linear partial-derivative operators upon it. From this general result, by specialization in different directions, theorems previously given by Kronecker, Faà di Bruno, and Scholtz may be obtained.

R. G. D. RICHARDSON,
Secretary

A CORRECTION

In the last line of page 140 of the March-April issue (this Bulletin, vol. 33 (1927)),

instead of $\{f(x)-[T_n(x)]\}^2$, read $\{f(x)-[T_n(x)]^2\}^2$.

THE FIFTY-FIRST REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-first regular meeting of the San Francisco Section of the Society was held at Stanford University on Saturday, April 2, 1927. The Chairman, Professor Allardice, presided. The total attendance was 45, including the following 29 members of the Society:

Allardice, Barter, Bernstein, Blichfeldt, Buck, Cairns, Cajori, Collier, Corbin, Eells, R. L. Green, M. W. Haskell, E. R. Hedrick, Hoskins, Hotelling, Glenn James, Vern James, D. H. Lehmer, D. N. Lehmer, S. H. Levy, W. A. Manning, Moreno, Neikirk, Noble, T. M. Putnam, Pauline Sperry, Stager, A. R. Williams, Wong.

It was decided to hold the next Spring meeting at Stanford University on Saturday, April 7, 1928.

Titles and abstracts of papers read at the meeting follow. Professor Bell's paper was read by title.

1. Dr. J. D. Barter: *On differential p-vectors.*

Those of the fundamental properties of differential p-vectors which appear pertinent to the theory of partial differential equations and of integral invariants are developed. Since a differential is characterized by the algebraic invariants of itself and of its vector derivative, considerable attention is given to these. Exception is taken to certain theorems and proofs of Gegenbauer, Schouten, and others, and corrections suggested. The nature of the automorphic transformations of differential p-vectors is specified and consequences indicated.

2. Dr. J. D. Barter: *Powers of determinants.*

It has seemed desirable to place on record certain determinant forms which may be given to the powers of determinants since the direct proof of the relations in question appears difficult and useful applications seem probable.

3. Dr. J. D. Barter: *On covariant differentiation.* Preliminary communication.

The two types of covariant differentiation, termed respectively albraic and vector, are derived and their mutual relation developed. Certain apparently novel results are obtained and applied to the theory of Jacobi's last multiplier and its generalization.

4. Professor E. T. Bell: *The arithmetic of logic.*

It is shown that the algebra L of logic is abstractly identical with certain parts of the arithmetic A of the rational integers. Formal equiva-

lents in L of the G.C.D., L.C.M., congruences, primes, arithmetical divisibility and the unique factorization law in A are developed. Lasker's concept of the residual in modular systems is necessary to complete the theory of congruences in L; it gives the equivalent in L of dividing (when possible) both members of a congruence in A by a common factor. There are two abstractly identical arithmetics of L, corresponding to the Peirce-Schröder dualism in L. The theory has applications to the classification of "doctrinal functions" (Keyser), or "system functions" (Sheffer). This paper will appear in the Transactions of this Society.

5. Professor W. D. Cairns: *Development in a system of approximately orthogonal functions.*

As is well known the terms of the expansion of $(1+1)^{2n}$, multiplied by a conveniently chosen factor, approach the ordinates of the curve $y=e^{-x^2/2}$ as n becomes infinite. This series of terms, and the principal parts of the successive differences formed therefrom, give polynomial functions which are dealt with in this paper. When combined by a summation analogous to the integration of orthogonal functions, these functions are shown to be orthogonal to within an error involving terms in $1/n$. A development of suitably restricted functions in terms of these functions is given, the approximation holding to within the order $1/n$.

6. Professor Florian Cajori: *The earliest arithmetic published in America.*

The author gives a description of the arithmetic of Pedro de Paz, a copy of which (partly in the original and partly in photostat reproduction) has been secured by the Library of the University of California. It was published in the city of Mexico in 1623. It is the earliest volume, on arithmetic exclusively, printed in America.

7. Professor Florian Cajori: *Unpublished letters of C. H. Schumacher and W. Struve to F. R. Hassler.*

The author gives an account of letters written by C. H. Schumacher and W. Struve to F. R. Hassler relating to astronomical publications, the design of improved geodetic instruments, and the procuring from Europe of expert copper engravers.

8. Professor Florian Cajori: *The logarithms of Napier.*

The author, with the aid of quotations from Napier, points out the invalidity of the argument against the accuracy of the formula Nap. $\log x = 10^7 \log (10^7/x)$, given on page 538 of the Proceedings of the National Academy of Sciences (September, 1926).

9. Professor E. R. Hedrick: *On certain expansive properties of neighborhoods in general spaces.*

Let us define as an expansive property of a region any property that holds true in any region provided it holds true in some region interior to the first one. Blumberg has recently presented a paper to the Society

(Philadelphia, Dec. 29, 1926) regarding extensions of the theorem that any derived set is closed. If a set is defined in any space that has the enclosable property, by means of an expansive property of the enclosing regions about each point of the set, it is shown in the present paper that the set is closed. Examples are cited, and applications to the theory, particularly to the validity of a Borel theorem, are mentioned.

10. Dr. Harold Hotelling: *Differential equations subject to error.*

A generalization of the theory of differential equations is obtained if we associate with every point, instead of a fixed direction, a frequency distribution of directions. In this way a frequency distribution can be be found for curves. It is argued in this paper that when an empirical law has been derived from a differential equation and statistical verification is sought, a more accurate test may often be applied to the differential equation itself than to its integral. The constants in the differential equation are then best determined first, the constants of integration afterward. These considerations are applied to the "logistic" hypothesis and the numerical work carried out for the population of the United States. Intercensal interpolation becomes a problem in the calculus of variations. For estimates by means of this theory the determination of probable errors is connected with the theories of random migration and heat conduction.

11. Professor Glenn James: *A theorem on monotonic decreasing functions.*

In evaluating certain monotonic decreasing series, need for further classification of such series arises. Toward meeting this need we formulate and prove the following theorem. *If* $\phi_0(x)$ *is positive and possesses derivatives to the nth order, those of even order being positive or zero, and those of odd order being negative or zero, and* $\phi_i(x)$ *denotes* $\phi_{i-1}(x)-\int_z^{z+b}\phi_{i-1}(z)dz$, $1\geqq b\geqq 0$, *then the same rule of signs applies to* $\phi_{i-1}(x)(i=1,2,3,\cdots,n)$ *and its first* $n-i$ *derivatives.* The theorem can be extended to functions of several variables.

12. Mr. D. H. Lehmer: *Note on the largest Mersenne number.*

The remarkable accuracy with which the famous statement of Mersenne in 1644 represents the facts concerning the primality of 2^n-1 is well known. The present author finds however that the largest number, $2^{257}-1$, given by Mersenne as a prime is really composite. This fact is revealed by the application of Lucas' test to the number. It was found that the 256th term of the recurring series 4, 14, 194, is not divisible by $2^{257}-1$. It follows then this 78-digit number is composite although it is impossible to say what its factors are. The test was undertaken in order to verify a similar calculation made by M. Kraitchik leading to the same general result. It has not been possible as yet to compare the two calculations.

13. Mr. D. H. Lehmer: *On the converse of Fermat's theorem.*

This paper appears in full in the present issue of this Bulletin.

14. Professor L. I. Neikirk: *A class of continuous curves defined by motion which has no tangent lines.*

The first example of a continuous curve without tangents was given by Weierstrass in 1872. Since then several others have been given. The most notable are those given by Peano, by Hilbert, by E. H. Moore, (*On certain crinkly curves*), and by W. F. Osgood, (*A Jordan curve of positive area*). The novel feature of the curve here discussed is its definition by linear and rotational motion which makes the lack of tangents geometrically evident. The paper includes an analytic proof of this fact.

15. Dr. A. R. Williams: *The quintic surface with two double straight lines.*

The quintic surface with two double straight lines is obviously universal. The plane system is composed of quintics having in common two double points and twelve simple points. There must be a relation between these fourteen points; for if taken arbitrarily they would impose eighteen conditions on the quintics. It is shown that they are the base points of a pencil of quartics that have a common tangent at the two which are the double points of the quintics. The class of the surface is in general thirty-four. There are twelve pinch points, six on each double line. There are, in general, thirteen lines which meet both double lines. Certain special cases are noted, wherein nodes or additional lines are produced by particular positions of the base points.

16. Dr. B. C. Wong: *Sextic surfaces with a double septic curve.*

By an involutorial quartic transformation by means of four hyperquadric surfaces in 4-space a plane σ is transformed into a two-dimensional variety $\sum_2^6$ of order 6. Projecting $\sum_2^6$ from any point z in S_4 on to a S_3 we obtain a surface of order 6 with a double septic curve. By taking z in positions related to $\sum_2^6$ or by having the base points, 10 in number, in σ take special relative positions, we obtain numerous special cases of the surface. This paper will appear in the Annals of Mathematics.

B. A. BERNSTEIN,
Secretary of the Section.

MATHEMATICS AND THE BIOLOGICAL SCIENCES*

BY H. B. WILLIAMS

We meet this afternoon to do honor to the memory of one whose influence on the development of scientific thought and achievement has been in many ways remarkable. The auspices under which this assembly has gathered is in itself an earnest of the regard in which the name of Josiah Willard Gibbs is held by the mathematicians of America. Forty years ago he delivered the vice-presidential address before the Section of Mathematics and Astronomy of the American Association for the Advancement of Science. The topic was *Multiple Algebra*.† His own contributions to this subject, particularly his development of the theory of dyadics, would be sufficient to establish without question his standing as a pure mathematician, while his vector analysis with its practical and convenient notation has been of no small service to the cultivators of mixed mathematics.

The soundness of his judgment in the field of physics is attested by the fact that he was among the first to take up and extend Maxwell's electromagnetic theory of light. In his obituary of Professor Gibbs it is remarked by Professor Bumstead‡ that these optical papers are noteworthy for the entire absence of special hypotheses regarding the connection between matter and ether. It seems to have been a characteristic trait with him to strip his problems of every unnecessary element before commencing extensive treatment of them.

* The Fourth Josiah Willard Gibbs Lecture, read at Philadelphia, December 28, 1926, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

† J. Willard Gibbs, *On multiple algebra*, Proceedings A. A. A. S., vol. 35 (1886), pp. 37–66. Also Gibbs, Collected Scientific Papers, vol. 2, p. 91.

‡ Bumstead, Henry A., *Josiah Willard Gibbs*, American Journal of Science, (4), vol. 16 (Sept. 1903). Also Gibbs, Scientific Papers, vol. 1.

Professor Hastings relates that Gibbs once replied to a complimentary remark about his work that if he had met with any success it was largely because he had found ways to avoid the mathematical difficulties! I am inclined to think that this was not intended altogether as a pleasantry. For although Gibbs met and overcame many serious mathematical difficulties in his work, he probably was well aware that he had avoided much unnecessary labor by this habit of first eliminating all but the bare essentials of his problem. Professor Bumstead remarks (loc. cit.) of these optical papers that it seems likely the considerations advanced in them would have sufficed firmly to establish the electromagnetic theory even if the experimental discoveries of Hertz had not supplied a more direct proof of its validity.

It is, however, his work in thermodynamics, particularly the great work, *On the equilibrium of heterogeneous substances*,* which has made the name of Josiah Willard Gibbs familiar in every part of the world where science is cultivated. The present year marks the fiftieth anniversary of the publication of the first part of this paper, the importance of which was not at once recognized. We shall hardly be surprised at this when we remember that the subjects treated are chiefly important to the chemist. Indeed this work may justly be said to have laid the foundation for theoretical chemistry. But in 1876, and for a good many years thereafter the number of chemists who had more than a nodding acquaintance with even the more elementary branches of mathematics was comparatively small. The paper is not only highly mathematical, but the methods employed are such as to make its reading a privilege open only to those who have had the foresight and industry to acquire a considerable mathematical armamentarium.

It is usually stated that the paper remained in comparative oblivion until its value was discovered by Professor Ostwald,

* Transactions of the Connecticut Academy of Arts and Sciences, vol. 3, pp. 108–248 and pp. 343–524. Also Gibbs, Scientific Papers, vol. 1, No. 3, pp. 56–353.

who, in 1891, translated it into German. However, I find in the presidential address of the late Lord Rayleigh,* given before the British Association at Montreal in 1884, the following comment, made in connection with his discussion of the dissipation of energy: "The foundations laid by Thomson now bear an edifice of no mean proportions, thanks to the labors of several physicists, among whom must be especially mentioned Willard Gibbs and Helmholtz. The former has elaborated a theory of the equilibrium of heterogeneous substances wide in its principles, and we cannot doubt, far reaching in its consequences." It thus appears that Lord Rayleigh had read and was fully cognizant of the importance of this paper prior to the Montreal meeting of 1884. His comment is at once a tribute to the genius of Willard Gibbs and a proof of the catholicity of his own scientific interests. In 1899 this paper was translated into French by LeChatelier and in 1906 it became more widely available to English readers through the publication of Professor Gibbs' collected Scientific Papers. To the influence of this paper it is largely due that chemistry, then an almost un-mathematical science, has so developed that the mathematical equipment now required by the student of chemistry differs but little from that which is requisite for the student of physics. The two subjects tend ever more and more to overlap.

Nor is it alone in theoretical chemistry that the influence of Gibbs is felt. It is hardly too much to say that chemical engineering and the whole vast edifice of chemical industry rest upon his work as their secure foundation.

The esteem in which Gibbs was held is reflected in many comments which may be found in the scientific literature. It is somewhat unusual to apply adjectives of commendatory character when quoting the work of another investigator and when this *is* done by a recognized master in his field, it signifies more than ordinary appreciation of the validity and

* Lord Rayleigh, Presidential Address, British Association Report, Montreal, 1884, pp. 1–23. Also Rayleigh, Scientific Papers, vol. 2, p. 342.

importance of the work. Quoting again from the works of Lord Rayleigh, in his paper, *On reflexion of light at a twin plane of a crystal*,* we find mention of "Professor Gibbs' excellent comparison of the elastic and electric theories of light with respect to the law of double refraction and dispersion of colors." Again in the paper on *Foam*† Lord Rayleigh alludes to "the masterly discussion of liquid films by Professor Willard Gibbs." In the review of Gibbs' collected papers by Jeans is the following: "The publication of the collected works of a great scientist serves the double purpose of forming a memorial to the genius of the writer and of increasing the usefulness of his labors by making his writings available to other workers. It is in connection with the second of these purposes rather than the first that one is inclined to consider the present volumes. They form, it is true, a memorial, and a fitting memorial to the greatness of Professor Gibbs' scientific work, but we feel that a memorial was hardly needed—his greatness is too well established and his work too well known for either to be enhanced at this stage by the publication of volumes of paper and ink."

In view of the fact that Professor Gibbs has exerted his greatest influence through the application of mathematical methods to the problems of a branch of science which, prior to his work, had not been regarded as a fertile field for theoretical cultivation, it seems not inappropriate that in meeting here to renew our recognition of the fundamental importance of his work we should turn our attention to the possibility of extending profitably the application of mathematical modes of thought to yet another field of scientific endeavor. To many the idea of applying mathematical methods to biological investigation may seem quite as unusual as did the idea of their application to chemistry at the

* Philosophical Magazine, vol. 26, p. 241; and Scientific Papers, vol. 3, p. 190.

† Proceedings of the Royal Institution, vol. 13 (1890), pp. 85–87. Also Scientific Papers, vol. 3, p. 359.

beginning of the last quarter of the nineteenth century. A tacit implication that the biological sciences are essentially un-mathematical subjects is conveyed by the titles under which the two sections of the Proceedings of the Royal Society of London are published. Section A is supposed to contain the mathematical and physical papers, while section B is the biological section. Notwithstanding this implication and the rather widespread impression that the problems of biology are too complicated or too indefinite, or both, to permit of mathematical treatment, certain of its questions have been so investigated and such studies still continue. Among those who have been successful in the application of mathematics to physiology may be mentioned no less a personage than Helmholtz. The mathematicians in my hearing all know of Helmholtz as a mathematician and the physicists know that he was regarded as one of the ablest physicists of his day, but because of his eminence in these fields it is easy to forget that he began his career as a physician and that his first scientific post was the professorship of physiology at Koenigsberg. From Koenigsberg he went to Bonn as professor of anatomy and physiology and from Bonn he was called to the professorship of physiology at Heidelberg. It was during these years that the ophthalmoscope was discovered and the great works on *The Science of Tone Perception* and on *Physiological Optics* were written. The latter work has recently been translated into English fifty years after its original publication. Although the translation was undertaken partly in commemoration of the centenary of his birth, it was welcomed more because this book is still, in many aspects of the subjects treated, the most authoritative work available. That it should remain a living book during all this period of intense scientific activity is probably due in no small measure to the precise habits of thought of the great investigator who wrote it. In this and in the work on sound will be found a wealth of examples of the application of mathematics to physiological problems. One might name a goodly number of men among the older

school of physiologists who made use of mathematics in various aspects of their work.

Shortly before the middle of the nineteenth century there began an epoch of great activity and productivity in organic chemistry. It was not long before the trend of thought and investigation in physiology began to show the influence of this activity in chemistry and physiological investigations have ever since tended to proceed more and more along chemical lines. As chemistry had not yet evolved into a mathematical science, it was not unnatural that physiologists should have tended away from mathematics in their training and there was a considerable period when this was largely true. There have always been a few individuals who either by chance or from personal predilection have brought to the study of physiology a considerable mathematical training and an occasional one who did so from conviction that this training is important for the most successful pursuit of his profession. In more recent years, and particularly since the advent of theoretical chemistry, there has been a growing tendency to reinstate mathematics as an essential part of the physiologist's intellectual equipment.

Looking now to the work of contemporaneous writers, the name of Gullstrand is pre-eminent in the field of physiological optics. One can hardly be said to be fully conversant with the present state of thought in this subject unless he has read the more important of Gullstrand's papers, * yet there are few ophthalmologists and not many physiologists who are able to read them and for the same reason that the chemists of Gibbs' day were unable to read his work. Gullstrand's

* Allvar Gullstrand, *Die reelle optische Abbildung*, Kungl. Svenska Vetenskapsakademiens Handlingar, vol. 41 (1906) No. 3; *Die optische Abbildung in heterogenen Medien und die Dioptrik der Kristal-linse des Menschen*, ibidem, vol. 43 (1908), No. 2; *Über Astigmatismus, Koma und Aberration*, Annalen der Physik, (4), vol. 18 (1905), pp. 941–973; *Tatsachen und Fiktionen in der Lehre von der optischen Abbildung*, Archiv für Optik, vol.1 (1907), p. 2; *Preparation of non spherical surfaces for optical instruments*, Kgl. Svenska Vetenskapsakademiens Handlingar, vol. 60 (1919), p. 155, abstracted in Zeitschrift für Instrumentenkunde, vol. 41 (1921), pp. 123–25.

investigations have already led to important practical results. I need only mention his ophthalmoscope, his special lenses for patients who have undergone operation for cataract and the slit-lamp microscopy of the living eye. The influence of his work would undoubtedly be more widespread were there more among the group of men interested who could read his papers. In one of the appendices to the recent translation of Helmholtz's *Physiological Optics*, Gullstrand has presented some of his conclusions freed to a great extent from the more difficult mathematics, but it is unfortunately not possible to present such a subject in the most convincing manner without the logical processes whereby the theory has been built up.

In most of our colleges and universities we require of science students, particularly those who are entering the graduate and professional schools, a reading knowledge of German and French. This is an admirable requirement and the reason for it is too well understood to need explanation. It is less well understood that not alone the chemist and physicist, but the biologist as well, must be able to read mathematical papers if he is not to be cut off from the possibility of understanding important communications in his own field of science. And the situation here is worse than it is in the case of inability to read a foreign language. For a paper in a foreign language may be translated, but in many cases it is impossible to express in ordinary language symbols the content of a mathematical paper in such a way as to convey a knowledge of the logical process by which the conclusions have been reached. The result may be accepted on faith, but this is repugnant to the scientifically trained mind.

Among recent outstanding contributions to physiology is the work of Professor A. V. Hill on the phenomena of muscle contraction which was crowned with the Nobel prize.* One might have thought it unlikely that new information of

* A. V. Hill and W. Hartree, *The four phases of heat-production of muscle*, Journal of Physiology, vol. 54 (1920), pp. 84–128.

fundamental importance would be unearthed in a field on which so much labor had already been bestowed. Our interest lies not so much in the ingenious methods devised by Hill as in the fact that they could have been devised and employed for the successful accomplishment of the purpose only by one who brought to the task the viewpoint and power of the skilled mathematician, trained in the concepts and methods of physics and physical chemistry. The absence of any formidable array of mathematical symbols from Professor Hill's papers might lead the casual reader to surmise that they are of the usual descriptive biological type. A more careful perusal will show that he has endeavored to present his results so far as possible without the use of mathematical symbols and phraseology in order that the papers may be intelligible to the physiologists of the present day. To do this he has been obliged to omit certain details, the nature of which can be readily surmised by one with some mathematical experience.* Professor Hill's training in mathematics and physics followed the well known traditions of Cambridge. That with this preparation, so unusual at the present time for a biologist, he should, from the very beginning of his productive work, have secured results of far reaching importance, carries a lesson which deserves to be well pondered by those whose duty it is to advise young men preparing for a lifework in biological research.

In any enumeration of the applications of mathematics to biological investigation the statistical treatment of biological data should occupy a prominent position. This field is at once one of the most important and one of the most treacherous into which the biological mathematician can

* Since the above was written, it has been possible to verify the accuracy of this surmise. Furthermore, though no mention of it is made in Dr. Hill's paper, the solution of an integral equation was part of the underlying mathematical structure essential to the success of the investigation. Although the inclusion of these details might have deterred many contemporaneous biologists from attempting to read the paper, it seems proper that the facts should be known because of the influence which they may, and should, exert on the rising generation.

venture. Gibbs frequently quoted in this connection a warning of Maxwell and I can do no better than repeat a passage taken from his obituary of Professor H. A. Newton, the astronomer. This was read by Gibbs before the National Academy of Sciences in 1897.* He says "This kind of investigation Maxwell has called statistical, and has in more than one passage signalized its difficulties. The writer recollects a passage of Maxwell which was pointed out to him by Professor Newton, in which the author says, that serious errors have been made in such inquiries by men whose competency in other branches of mathematics was unquestioned." Gibbs was himself a master of this type of investigation and the biological sciences are fortunate in having secured the services of a former pupil and later distinguished colleague of Gibbs, Professor E. B. Wilson, who is now connected with the Harvard Medical School. Time will not permit a lengthy enumeration of the problems which have been approached by this method. Those who are interested will find ample references in the works of Professor Karl Pearson and in those of Professor Raymond Pearl. A recent interesting achievement in the field of statistical biology is the development by Mr. Arne Fisher† of a method of constructing from biological data a series of frequency curves for the death rates at various ages from certain groups of causes. These he has been able to combine into a compound frequency curve from which may be predicted the expected deaths at various ages in an entire population without knowledge of the total number of lives exposed to risk. The resulting curves have been shown to be in good agreement with curves established by the usual actuarial methods for the same populations. The method has been

* J. Willard Gibbs, *Hubert Anson Newton*, American Journal of Science, (4), vol. 3 (1897), pp. 359–376. Also Gibbs, Scientific Papers, vol. 2, pp. 268–284.

† Arne Fisher, *Frequency Curves*, New York, The Macmillan Co., 1922; *Note on a new method of construction of mortality tables when the number of lives exposed to risk is unknown*, Skandinavisk Aktuarietidskrift, 1925 pp. 163–215.

subjected to vigorous attack by actuaries unfamiliar with the biological principles involved, but there can be little doubt of the essential soundness of the procedure and its applicability is by no means limited to actuarial studies.

The use of statistical methods in biological studies has hitherto been confined largely to such matters as are included in the term biometry. Among the various items would be such as birth and death rates, incidence of morbidity, curves of population growth, correlations between various classes of biometric data and the like. These subjects are of great importance, an importance which I recognize fully at the same time that I feel impelled to offer the suggestion that statistical mathematics will perform its greatest service for the biological sciences in a very different field, a field as yet practically untouched. It may even be premature to suggest this application in the present state of our quantitative biological data. In a mathematical study of any system in which it is possible to measure the total integrated changes of energy or of material state, but impossible, on account of its complexity, the smallness of its parts or their inaccessibility to our methods of investigation, to know in detail all of the elements which contribute to the total change, we must have recourse to the statistical method. The brilliant success with which Maxwell has applied this method to the kinetic theory of gases and its power in the hands of Gibbs in investigating the energetics of chemical reactions, entices us toward the thought—the hope, perhaps—that one day it may come to be applied to such studies as the energetics of secretion and other biological processes. If the whole complicated mammalian organism operates in accordance with the principle of the conservation of energy, as experiments with the respiration calorimeters show it to do, it seems altogether likely that the minute details of the energy exchanges will be found to obey the same laws that hold in the world of the non-living. Only in the living organismone must reckon with the effect of the structure. This seems to be the chief peculiarity of living things and in any attempt

to apply the methods of chemistry and physics to the problems of biology one must keep constantly in sight the fact that structure and its consequences is an integral part of the problem. It is an item which seems to increase the difficulties enormously. Much data have already been accumulated. It is possible that some further information regarding *structure* is needed before the next step can be taken, possibly something about finer structure than we have been able to study with the microscope hitherto.

My colleague, Professor E. L. Scott, has recently carried out an investigation on the *number* of experiments which must be performed before the results acquire real significance. Such a study is partly mathematical, but there must enter into it experimental data, since the answer is intimately bound up with the uniformity with which the experimental conditions can be controlled and repeated and the certainty with which the experimenter can know that he has controlled the conditions within any specified limits of uniformity. In three series of observations, each in a different biological class, Dr. Scott finds that the number of experiments required for reasonable certainty is of the order of fifty. It is too early to predict whether this order of magnitude applies to any considerable number of types of biological experiment. However, it raises at once the question whether in any short series of biological observations the variations which occur can be regarded as due necessarily to experimental procedure. The variability which Professor Scott has found in the three types studied would seem to make it incumbent upon any investigator who desires to draw important conclusions from short series of observations to show that in the type of work he is doing, the variability is small enough to justify his procedure. In order to do this he must run at least one much longer series of experiments, since at times the larger variations may fail to occur in the first five or ten observations. A small dispersion in a series of five experiments may be sadly misleading.

If we are to begin to make biology an exact science, one of the first requisites will be reliable data. Familiarity with the theory of errors and a critical attitude in drawing conclusions from measurements are characteristics of the men who cultivate those sciences we now regard as "exact." We shall have taken an important step when biologists regard their results in the same critical light. Teachers of mathematics can probably further this desirable situation by explaining to students, who are known to contemplate a career in one of the biological sciences, the importance of a knowledge of the theory of measurements.

From time to time physiologists are constrained to enter upon mathematical investigations not so directly related to biological processes themselves. It sometimes happens that correct interpretation of experimental results requires a study of the theory of the instruments used in making the measurements. At times these inquiries have resulted in substantial improvements in methods of work and design of instruments. Notable examples are to be found in the work of Professor Einthoven in the field of electrophysiology and in that of Professor Otto Frank in hemodynamics.

Physiology is sometimes defined as the physics and chemistry of living things. While this definition resembles most definitions in that it is somewhat lacking in completeness, it serves to direct attention to the particular phases of physiology in which it appears most probable the greatest progress of the immediate future will be made. This is not my personal view alone. A distinguished zoologist recently voiced quite similar opinions in my hearing. That the plant physiologists have reached similar conclusions is evidenced by the attitude of Hampton and Gordon, * who in a recent issue of Science, plead for better preparation in mathematics, physics and chemistry as a foundation for research in plant physiology. They quote Lepeschkin as holding the view that plant physiology must develop hand in hand with

* H. C. Hampton and S. C. Gordon, *A suggested course in plant physiology*, Science, vol. 64 (1926), pp. 417–419.

physics and chemistry. I can see no escape from the conclusion that the physiologist of the immediate future must be to a considerable extent a chemist and physicist and these he can be only by building from a suitable mathematical foundation. Nor is this a hopeless prospect. Textbooks of physics of seventy-five years ago were much larger than at present. This in spite of the enormous additions since made to our knowledge of the subject. But these older books were voluminous because of minute descriptions of phenomena which we now recognize as what a mathematician would call particular cases, comprehended under broad general principles. Chemistry has also grown greatly in detail, but study of the phenomena of radioactivity and the discovery of the isotopes have already done much to simplify its concepts and we may look forward confidently to a time when our successors will be able to learn the broad principles of chemistry much more readily than can be done at present.

The ultimate object of all scientific study is to enable us to predict that in certain circumstances certain events will happen. The old saying, "History repeats itself," is tantamount to recognition that even so complicated a system as large groups of human beings reacts similarly in similar circumstances. We are not yet able to predict in biological matters to the same order of precision as in physics and chemistry, but that is no proof of lack of precision in biological reactions. Rather it points to our inability to control conditions and to recognize small, but significant, points of difference. Perhaps we may look forward to a day when the history of generalization in chemistry and physics may repeat itself in the biological sciences.

Cognate with the problem of securing a more general appreciation of the value of mathematical methods in biological investigation is that of overcoming a certain feeling of aloofness from these methods which in many people amounts almost to a pathological phobia. That the study of mathematics presents difficulties no one will deny, but that the difficulties are so great as many imagine them to be is

certainly *not* true of those branches most applicable to the problems of the physical world as we see them today.

There seems to be a rather widespread opinion that some individuals are endowed with an innate capacity for mathematical thinking which is quite as definitely lacking in others. There is probably *some* basis for this opinion. If we think of natural adaptations of more evident type we can find plenty of analogies. For example, no amount of training would enable a heavy Clydesdale horse to compete successfully against a thoroughbred in a running race. Nevertheless the Clydesdale *can* run and given a little more time would cover the same ground as the animal better adapted for speed. I fear that many students, quickly fatigued by the sustained mental effort requisite to follow the reasoning in a new branch of mathematics, conclude too readily that they are of the group who have not an hereditary capacity for this discipline. There are probably not a few professional mathematicians whose inherited capacity is actually less than that of men who have quit the subject in despair. The man of high courage and determination often succeeds in spite of heavy handicaps, while in every field of human endeavor we see failure, often after an auspicious beginning, where these important qualities are lacking. Where courage and industrious habit is combined with large natural capacity we may expect an unusual measure of success, but I am inclined to think that there are few men—or women—whose intellectual capacity is sufficient to permit their successful cultivation of any branch of science, who could not, if they felt an urgent need for it, obtain a sufficient mastery of those branches of mathematics most requisite for the study of the natural sciences.

Before any person of sound judgment will undertake a difficult task he must first be convinced that it is worth doing and that he has a reasonable chance of success if he undertakes it. I have known more than one man of mature years with very meager training in elementary mathematics who, finding himself checked by this lack, has acquired the

desired facility by study, aided only by books, and usually crowded into the routine of a busy life. The rapidity with which such a student often progresses depends, it seems to me, very greatly upon the maturity of his mental processes. He knows exactly why he wants the information; he sees beyond the task of the moment and has from the beginning a rather clear conception whither his study is to lead him. Furthermore he is likely to have attained to a certain measure of self-discipline and is able to keep his thoughts from wandering. This requires no effort at all once a real interest in the subject has been awakened. I have known of one instance where such auto-instruction has been attended with marked success in a man who as a youth regarded himself and was regarded by his teachers as one of those who have no capacity for advanced mathematical study. When it becomes necessary to spend the time of maturer years in thus acquiring what might have been secured in youth, the process always involves a certain waste. The individual has been deprived for a number of years of an intellectual aid which he might have had at his service, and the time of his maturity might be better employed in occupations more suited to his riper experience. It seems that the lesson to be derived is that the awakening of a very real interest on the part of the student in the subject of his study is quite as necessary for his success as is the possession of innate capacity.

One aspect of mathematical study which it seems does much to discourage the beginner is the smooth and finished character of every demonstration he meets. Realizing keenly that he could hardly hope to carry through any such perfectly finished logical reasoning on an original problem, he becomes discouraged. He thinks that these textbook demonstrations are models of the way in which mathematicians do their work. Even the artifices employed awaken mingled feelings of admiration for the mind that can conceive and apply them with such wonderful facility, and despair that he will ever be able to match this performance when confronted with an original problem with no one at hand to suggest the

manner of approach. The wise teacher can do a great deal at this stage for his encouragement. He may be told that even a finished mathematician usually writes down on his work-paper much more than he subsequently permits to be printed. Also that not infrequently when a result has been reached by devious ways and after expenditure of much time and labor, it is afterward seen to be capable of direct approach. He will thus become aware that the smooth and concise demonstration is not necessarily conceived in that finished form. I wonder if, for its educational value, it might not be better at times to let students see how some of these results which can be expressed so compactly and neatly when we know just how to do it, were actually reached originally. Beginners can hardly be expected to go to the primary sources, but the teacher who is familiar with the original literature can often perform a real service to his students by bringing to their attention passages from the older authors who, writing at a time when knowledge of their subject had not yet been cast into its present condensed form, sometimes shed light on points which give difficulty in the modern standard treatises.

I am reminded of a passage in Maxwell's *Electricity and Magnetism*.* Referring to the brilliant demonstration by Ampère of the laws of action between electric currents, he says;

"The experimental investigation by which Ampère established the laws of the mechanical action between electric currents is one of the most brilliant achievements in science. The whole, theory and experiment, seems as if it had leaped full grown and full armed from the brain of the 'Newton of electricity'. It is perfect in form, unassailable in accuracy and it is summed up in a formula from which all the phenomena may be deduced, and which will always remain the cardinal formula of electrodynamics. The method of Ampère, however, though cast into an inductive form, does not allow us to

* J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Oxford, Clarendon Press, 1873, vol. 2, p. 162, § 528.

trace the formation of the ideas which guided it. We can scarcely believe that Ampère discovered the law of action by means of the experiment which he describes. We are led to suspect, what indeed, he tells us himself, that he discovered the law by some process which he has not shewn us, and that when he had afterwards built up a perfect demonstration, he removed all traces of the scaffolding by which he raised it. Faraday on the other hand, shews us his unsuccessful as well as his successful experiments, and his crude ideas as well as his developed ones, and the reader, however inferior to him in inductive power, feels sympathy even more than admiration, and is tempted to believe that, if he had the opportunity, he too would be a discoverer. Every student therefore should read Ampère's research as a splendid example of scientific style in the statement of a discovery, but he should also study Faraday for the cultivation of a scientific spirit, by means of the action and reaction which will take place between newly discovered facts and nascent ideas in his own mind." Although written by a physicist about a question in physics, it seems to me similar considerations apply to mathematics. If the student can be made to realize at the outset of his work that the beautiful edifice he admires has not arisen by some species of black magic out of nothing, but was originally cumbered by the scaffolding and debris of construction, he will learn that he must expect to accumulate a certain amount of debris when he undertakes to produce something of an original character.

A difficulty which not infrequently stops the novice in reading mathematical papers lies in his attempting to read from formula to formula, thinking that because the writer has written them in immediate succession, he should be able to see the derivation at once without the necessity of performing any work with paper and pencil. A reader of long experience with the subject under discussion may be able at times to do this, but the student should be cautioned in every case where it is not "evident" to start manipulating with paper and pencil the last formula he understands until the

next succeeding one has been derived. In writing papers which will probably be read only by professional mathematicians, authors not infrequently omit so many intermediate steps in order to condense their papers that the filling in of the gaps even by industrious use of paper and pencil may become no inconsiderable labor, especially to one approaching the subject for the first time. In reviewing the Electrical Papers of Oliver Heaviside, Professor FitzGerald remarks:* "In his most deliberate attempts to be elementary, he jumps deep double fences and introduces short cut expressions that are woeful stumbling-blocks to the slow-paced mind of the average man. . . ." Those who write for biological readers of the present time should bear in mind that the saving of effort which the insertion of a few, possibly not altogether necessary, formulas will effect may make all the difference between success in comprehending the paper and utter discouragement. Since papers are printed to be read and to get results, the saving of a little paper and ink at the expense of great loss of clarity is surely no gain.

It may not be wholly uninteresting to the mathematicians and possibly may shed light on the value of mathematics for those less familiar with its possibilities to advert briefly to some considerations of the subject colored by a physiological background. It is well known that man is set apart from other animals by reason of his greater intellectual powers. Wherein is the source of these powers? There are structural differences between the brains of the most highly developed of the lower animals and that of man, but when one considers the great disparity of function he is struck by the relative smallness of the structural difference. There is evidence, which I cannot stop to present, that the memory of the lower animals is to a great extent, if not entirely, a memory of particular concrete instances. Through the development of language man has substituted for these, abstract and general symbols which represent any object or event of a

* G. F. FitzGerald, *Heaviside's electrical papers*, Electrician, August 11, 1893. Also G. F. FitzGerald, Collected Scientific Writings, No. 61, p. 293.

given class. The phenomena of aphasia give evidence that certain limited parts of the brain are concerned with the language function and that thinking is ordinarily done in language symbols, usually in sound symbols. It is evident how much more easily the processes of thought can be extended in general symbols than would be the case if all thought had to deal with memories of particular and concrete objects. Now mathematics is both a body of truth and a special language, a language more carefully defined and more highly abstracted than our ordinary medium of thought and expression. Also it differs from ordinary languages in this important particular: it is subject to rules of manipulation. Once a statement is cast into mathematical form it may be manipulated in accordance with these rules and every configuration of the symbols will represent facts in harmony with and dependent on those contained in the original statement. Now this comes very close to what we conceive the action of the brain structures to be in performing intellectual acts with the symbols of ordinary language. In a sense, therefore, the mathematician has been able to perfect a device through which a part of the labor of logical thought is carried on outside the central nervous system with only that supervision which is requisite to manipulate the symbols in accordance with the rules. It is not even necessary during the intermediate mathematical steps to inquire what meaning is to be attached to the groupings of symbols. The thought which I have here in mind has been well expressed by Maxwell,* in referring to the work of Lagrange. Maxwell says: "The aim of Lagrange was to bring dynamics under the power of the calculus. He began by expressing the elementary dynamical relations in terms of the corresponding relations of pure algebraical quantities, and from the equations thus obtained he deduced his final equations by a purely algebraical process. Certain quantities (expressing the reactions between the parts of the system called into play by

* J. C. Maxwell, *Electricity and Magnetism*, Ed. of 1873, vol. 2, p. 184, § 554.

its physical connexions) appear in the equations of motion of the component parts of the system, and Lagrange's investigation, as seen from a mathematical point of view, is a method of eliminating these quantities from the final equations. In following the steps of this elimination the mind is exercised in calculation, and should therefore be kept free from the intrusion of dynamical ideas. Our aim, on the other hand, is to cultivate our dynamical ideas. We therefore avail ourselves of the labours of the mathematicians, and retranslate their results from the language of the calculus into the language of dynamics, so that our words may call up the mental image, not of some algebraical process, but of some property of moving bodies." These remarks of Maxwell would apply just as truly had the particular dynamical phenomena he had in mind been those of protoplasm.

One often hears it remarked that no new knowledge can be obtained by mathematical processes. The same may be as truly said of other modes of thought in the only sense in which it is true of mathematics. Thought cannot begin until there is material to think about, but the same material is presented alike to the animal, the savage, the child, and the scholar. No one who has given the matter serious thought will deny that the intellectual processes, whether mathematical or otherwise, do discover important truth in handling the material presented by the external world. It is the important *relationships* between the facts of observation which are discovered by thought and form the basis of our intellectual lives. Of all the aids to thought that man has devised the most powerful is mathematics. So long as thinking must be carried on solely in memories of particular things or of their corresponding sound symbols it is difficult to be certain that one has made a reasonably complete survey of the relationships between the facts, that he has realized all of the implications contained in them. Formulation of the results in a mathematical expression may not quite adequately represent them. Usually it represents a system simplified in order that the corresponding mathematical expressions may not be

beyond our powers of manipulation. If the behavior of the simplified system can be regarded as sufficiently similar to that of the real one which we wish to study, we have secured a powerful aid in seeking the implications of the facts which are at our disposal. Whenever any of the natural sciences has been able so to formulate its observed facts, progress in that branch has been accelerated in a manner so remarkable as to leave no room for doubt of the efficacy of the mathematical method. The greatest need of the biological sciences today is to discover amongst the wealth of empirical facts those relationships which are most significant; those whose recognition will result in the transfer of long lists of particular items to a few general classes, thus simplifying our views regarding them.

The physiologist deals with those aspects of life which are subject to observation and measurement by the same instruments and methods that are used in the study of non-living matter. If there be vital phenomena of a transcendent character, these do not concern the physiologist as a physiologist. So soon as the physical and chemical data, including the structural relationships, have been established with reasonable certainty, the work of the mathematician may begin. There is still much uncertainty about some of the data and their improvement by critical study is perhaps the first requisite. In this we shall need some mathematical assistance also. So soon as it becomes possible to construct biological theories cast in mathematical form we may look for rapid progress. Experiment will then be guided along those lines where it is most apt to yield new knowledge of importance. This has been the history of physics and chemistry. The evolution of chemistry into an exact science has been so recent that many of us have actually witnessed the events. We can hardly doubt that a similar outcome for the biological sciences will one day become a matter of history also.

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ON A GENERALIZATION OF THE SECULAR EQUATION*

BY JAMES PIERPONT

1 *Introduction.* The equation we wish to consider is

$$H_n(x) = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix} = 0. \tag{1}$$

Here $a_{ij}=a_{ji}$, when $i \neq j$; while

$$\begin{aligned} a_{ii} &= \alpha_{ii} - x, \quad \text{for } i = 1,2, \cdots, r, \\ &= \alpha_{ii} + x, \quad \text{for } i = r+1, r+2, \cdots, n. \end{aligned}$$

The a_{ij} and α_{ii} are real, and $H(0) \neq 0$. If $r=n$, (1) is the secular equation which it will be convenient to denote by $L_n(x)=0$. When $r=n-1$ the equation (1) plays a fundamental role in classifying quadric surfaces in n-way hyperbolic space. Let us set $n-r=s$ and call $\sigma = |r-s|$ the *signature* of (1). We have then the

THEOREM I. *The number of real roots of $H_n(x)=0$, counting their multiplicity, is not less than its signature.*

This is a corollary of a theorem to which F. Klein calls especial attention (Mathematische Annalen, vol. 23 (1884), p. 562). The proof there given rests on the theory of elementary divisors.† We give here a very simple proof which is a modification of H. Weber's proof that the roots of the secular equation $L_n(x)=0$ are all real.‡ Weber's proof as we shall see, is complicated by his belief that it is necessary to show that

$$L_n'(x) = -\sum_i \frac{\partial L_n}{\partial a_{ii}}, \qquad (i = 1,2, \cdots, n).$$

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* Presented to the Society, December 29, 1926.

† See T. J. Bromwich, *Quadratic Forms*, Cambridge Tracts, No. 3 (1906), p. 69.

‡ H. Weber, *Algebra*, vol. 1, 1898, pp. 307–310.

2. *Proof of the Theorem.* We turn now to the proof of the above Theorem I. Consider the sequence

$$H_n, H_{n-1}, H_{n-2}, \cdots, H_1, H_0 = 1, \tag{2}$$

where H_k is the determinant of degree k in x obtained by deleting the last $n-k$ rows and columns of (1). For the moment we suppose that no two of the H's vanish for the same x. They are connected by the relations

$$\begin{aligned} H_n H_{n-2} &= H_{n-1}\phi_{n-1} - \psi_{n-1}^2, \\ H_{n-1} H_{n-3} &= H_{n-2}\phi_{n-2} - \psi_{n-2}^2, \\ &\cdots\cdots\cdots\cdots \\ H_2 H_0 &= H_1\phi_1 - \psi_1^2, \end{aligned} \tag{3}$$

where ϕ, ψ are polynomials in x.

Merely for completeness let us show how these relations are obtained, the first for example. Let A_{ij} be the minor of a_{ij} in (1); set $\nu = n-1$. Then

$$B = \begin{vmatrix} A_{\nu\nu} & A_{\nu n} \\ A_{n\nu} & A_{nn} \end{vmatrix} = H_{n-1} A_{\nu\nu} - A_{n\nu}^2 . \tag{4}$$

But $H_n \cdot B = H_{n-2} \cdot H_n^2$. Hence, if $H_n \neq 0$, $B = H_{n-2} \cdot H_n$. This with (4) gives

$$H_n H_{n-2} = H_{n-1}\phi_{n-1} - \psi_{n-1}^2.$$

This relation holds also if $H_n = 0$, as continuity considerations show.*

The equations (3) show that when $H_k = 0$, H_{k+1}, H_{k-1} have opposite signs.

We now consider the signs of the sequence (2). Suppose $H_n = 0$ for $x = a$. Then in a sufficiently small interval δ about $x = a$, H_n changes its sign, while none of the other terms in (2) do. Thus as x passes through δ the sequence (2) gains or loses one variation of sign. On the other hand when x passes through a root of $H_{n-1}, H_{n-2}, \cdots$ no variation is gained or lost as (3) show. For $x = +\infty$ there are r

* See Weber, loc. cit., p. 113; or Kowalewski, *Determinantentheorie*, p. 83.

variations of sign in (2); for $x=-\infty$ there are s, thus $H_n(x)=0$ has at least σ real roots.

We now consider the general case that the sequence (2) has common roots. With Weber we may dispose of this case as follows. Suppose e.g. that H_k, H_{k-1} have common roots. We vary the terms a_{ij} of H_k not in H_{k-1} by small amounts numerically less than some η, so that H_k, H_{k-1} do not have common roots.

In this way we may replace (2) by another sequence

$$K_n, K_{n-1}, K_{n-2}, \cdots, K_1, K_0 = 1 \tag{5}$$

no two of which have a common root. The roots of $K_n=0$ differ from those of $H_n=0$ by an amount as small as we please, for sufficiently small η, moreover the signs of corresponding elements of the sequences (2), (5) are the same for an x for which no element of (2) vanishes. As Theorem I holds for (5), it must hold for (2).

THEOREM II. *The roots of the secular equations are all real.*

For in this equation $s=0$; hence $\sigma=n$.

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A GENERALIZED TWO-DIMENSIONAL POTENTIAL PROBLEM

BY J. R. CARSON

It may be shown that the solution of the problem of electromagnetic wave propagation along a system of straight parallel conductors can be reduced to the solution* of two subsidiary problems: (1) a well known problem in two-dimensional potential theory; and (2) a generalization of the two-dimensional potential problem which is believed to be novel. The generalized problem is believed to possess suffi-

* Subject to certain restrictions to be discussed in a forthcoming paper.

cient mathematical interest, in addition to its technical importance, to justify its publication in this note.

The two-dimensional potential problem will first be stated in order to exhibit its relation to the generalized problem:

Let there be n closed curves in the xy-plane and let a function $\Phi(x, y)$ satisfy Laplace's equation in two dimensions everywhere outside these curves; thus

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi = 0. \tag{1}$$

On the jth curve ($j=1, 2, \cdots, n$), let

$$\frac{\partial}{\partial \tau}\Phi = 0, \quad \int \frac{\partial}{\partial n}\Phi d\tau = Q_j, \quad (j = 1,2, \cdots, n). \tag{2}$$

In equations (2), n and τ are the vectors normal and tangential to the jth curve respectively and Q_j is a specified real constant.

The determination of the potential function Φ in terms of $Q_1, \cdots, Q_n$ and the geometry of the field is a well known problem in the theory of the potential for which very general methods of solution are available. The solution is of the form

$$\Phi = \phi_1(x,y)Q_1 + \cdots + \phi_n(x,y)Q_n, \tag{3}$$

the coefficients $\phi_1, \cdots, \phi_n$, themselves two-dimensional potentials, being determined by the geometry of the field.

The generalized problem, which is the subject of this note, may be stated as follows.

Let there be n closed curves in the xy-plane, and let a function F satisfy the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F = 0 \tag{4}$$

everywhere outside these curves.

Inside the jth curve let a function E_j satisfy the equation

$$(5) \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)E_j = \nu_j^2 E_j, \qquad (j = 1,2, \cdots, n),$$

where ν_j^2 is a given parameter, in general complex.

On the jth curve let F and E_j be connected by the equations

$$(6) \quad \frac{\partial}{\partial \tau}F = -\frac{\partial}{\partial \tau}E_j, \quad \frac{\partial}{\partial n}F = -\mu_j\frac{\partial}{\partial n}E_j, \ (j = 1,2, \cdots, n).$$

Here, as above, n and τ are the vectors normal and tangential to the jth curve respectively, and μ_j is a given real parameter.

Finally

$$(7) \qquad \int \frac{\partial}{\partial n}E_j d\tau = -\int \frac{\partial}{\partial n}F d\tau = J_j, \qquad (j = 1,2, \cdots, n),$$

where J_j is a specified constant, in general complex. The integration is carried around the contour of the jth curve.

We require the determination of F outside the curves and E_j inside the jth curve in terms of $\lambda_j, \mu_j, \nu_j, J_j$ and the geometry of the field, in the form

$$(8) \qquad \begin{aligned} F &= f_1(x,y)J_1 + \cdots + f_n(x,y)J_n, \\ E_j &= e_{j1}(x,y)J_1 + \cdots + e_{jn}(x,y)J_n. \end{aligned}$$

This problem has been solved for the practically important case where all the curves are circles by expanding E_j as a Fourier-Bessel series and identifying the constants of the expansion by a process of successive approximations. As yet, however, no serious study has been made of the general problem. Its analogy with the potential problem suggests, however, that extensions of the well known methods there available should be successful in the solution of the generalized problem.

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A GENERALIZATION OF WARING'S THEOREM ON NINE CUBES*

BY L. E. DICKSON

THEOREM. *Every positive integer p can be expressed as a sum of seven cubes and the double of a cube, the cubes being positive or zero integers.*

On the basis of known tables, this theorem holds for $p \leqq 40{,}000$ as shown by the writer in the American Mathematical Monthly for April, 1927. The further empirical theorems of that paper will not be discussed here.

We shall here prove the above theorem for all sufficiently large integers p. The proof is analogous to that employed by Landau† in proving his theorem that every sufficiently large integer is a sum of at most eight positive integral cubes.

Let r be the real ninth root of $4/3$. The number of the primes $\equiv 2 \pmod 3$ which exceed x and are $\leqq rx$ is known to increase indefinitely with x. As x we choose the first radical in (1). Hence for all sufficiently large integers n, there exist at least ten primes p such that

$$(1) \qquad (n/12)^{1/9} < p \leqq (n/9)^{1/9},$$

$$(2) \qquad p \equiv 2 \pmod 3.$$

The product of the ten primes exceeds $(n/12)^{10/9}$ and hence exceeds n if $n > 12^{10}$. Hence not all of the ten are divisors of n.

To give a numerical illustration, take $n = 9m^9$. Then (1) becomes $m/r < p \leqq m$. For $m = 6000$, $m/r = 5811.2$, and the primes p satisfying also (2) are

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* Presented to the Society, April 15, 1927.

† Mathematische Annalen, vol. 66 (1909), pp. 102–5. Reproduced in his *Verteilung der Primzahlen*, vol. 1, 1909, pp. 555–59.

$$5813,\quad 5843,\quad 5849,\quad 5861,\quad 5867,\quad 5879,$$
$$5897,\quad 5903,\quad 5927,\quad 5939,\quad 5981,\quad 5987.$$

The third prime just exceeds m'/r when $m'=6038$. Hence the last ten primes of our list serve for every m between 6000 and 6038 inclusive.

Let therefore p be a prime not dividing n such that (1) and (2) hold. Then

$$9p^9 \leqq n < 12p^9.$$

A slight modification of Landau's proof of his first lemma shows that if p is an odd prime satisfying (2), every integer not divisible by p is congruent modulo p^3 to the double of a cube. Hence there are integers δ and M satisfying

$$n - 2\delta^3 = p^3M, \quad 0 < \delta < p^3.$$

Then

$$9p^9 - 2p^9 \leqq n - 2p^9 < n - 2\delta^3 = p^3M, \quad p^3M < n < 12p^9.$$

Cancelling the factors p^3, we have

$$7p^6 < M < 12p^6.$$

The further discussion by Landau* applies here unchanged and shows that n is the sum of $2\delta^3$ and seven integral cubes $\geqq 0$. We may replace $2\delta^3$ by $k\delta^3$; $k \geqq 1$.

THE UNIVERSITY OF CHICAGO

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* His condition for $M_2>0$ is satisfied if $p\geqq 10$. Since the largest γ is 22, we obtain the milder condition $p\geqq 5$. Hence n exceeds 13 million.

AN ELEMENTARY PROOF BY MATHEMATICAL INDUCTION OF THE EQUIVALENCE OF THE CESÀRO AND HÖLDER SUM FORMULAS*

BY TOMLINSON FORT

For brevity, notation and terminology in this note are generally not explained. It is believed that they will be clear in all instances to any probable reader.

THEOREM. *The Hölder and Cesàro methods of summation are equivalent.*

PROOF. Let $C_n^{(r)}$ represent the Cesàro sum of order r to n terms:

$$(1)\qquad C_n^{(r)} = s_0\frac{r}{r+n} + \cdots + s_k\frac{r(n-k+1)\cdots n}{(r+n-k)\cdots(r+n)} + \cdots + s_n\frac{r(n!)}{r\cdots(r+n)}.$$

We readily verify that $C_n^{(r)}$ satisfies the equation

$$(2)\qquad (n+r+1)\,C_n^{(r+1)} - nC_{(n-1)}^{(r+1)} = (r+1)C_n^{(r)},$$

which may be written in the form

$$\Delta\{nC_{(n-1)}^{(r+1)}\} + r\,C_n^{(r+1)} = (r+1)C_n^{(r)},$$

or

$$(3)\qquad (n+1)C_n^{(r+1)} + r\sum_{n=0}^{n} C_n^{(r+1)} = (r+1)\sum_{n=0}^{n} C_n^{(r)}.$$

By solving (2) we get the following, as is easily verified:

$$(4)\qquad C_n^{(r+1)} = \frac{n!}{(r+2)\cdots(r+n+1)}\sum_{n=0}^{n}\frac{(r+1)\cdots(r+n)}{n!}C_n^{(r)} = \frac{(r+1)!}{(n+1)\cdots(n+r+1)}\sum_{n=0}^{n}\frac{(n+1)\cdots(n+r)}{r!}C_n^{(r)}.$$

*Presented to the Society, December 29, 1926.

Let $H_n^{(r)}$ represent the Hölder sum to n terms of order r. We establish by mathematical induction the following fundamental formula.

$$
\begin{aligned}
H_n^{(r)} &= k_0^{(r)} C_n^{(r)} + k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \\
&+ k_2^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} + \cdots \\
&+ k_{r-1}^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \cdots \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} ,
\end{aligned} \tag{5}
$$

where

$$k_0^{(r)} + k_1^{(r)} + \cdots + k_{r-1}^{(r)} = 1.$$

If we assume (5), we find

$$
\begin{aligned}
H_n^{(r+1)} = \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)} &= k_0^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \\
&+ k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \\
&+ k_2^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} \\
&+ \cdots + k_{r-1}^{(r)} \frac{1}{r+1} \sum_{n=0}^{n} \frac{1}{n+1} \cdots \sum_{n=0}^{n} C_n^{(r)} .
\end{aligned} \tag{6}
$$

Substitute for

$$\frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)}$$

in each sum of (6), its value from (3). Take for example

$$
\begin{aligned}
k_1^{(r)} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r)} &= k_1^{(r)} \frac{1}{r+1} \cdot \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r+1)} \\
&+ k_1^{(r)} \frac{r}{n+1} \frac{1}{n+1} \sum_{n=0}^{n} \frac{1}{n+1} \sum_{n=0}^{n} C_n^{(r+1)} .
\end{aligned} \tag{7}
$$

We notice that

$$k_1^{(r)} \cdot \frac{1}{r+1} + k_1^{(r)} \frac{r}{r+1} = k_1^{(r)}.$$

We then notice that we have an expression of the same form as (5), but with r replaced by $(r+1)$. As $H_n^{(1)} = C_n^{(1)}$, proof of the formula follows by induction.

We next prove in a similar manner the formula

$$(8)\quad C_n^{(r)} = h_0^{(r)} H_n^{(r)} + h_1^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-1)}{(r-1)!} H_n^{(r)} + h_2^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n}\sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r)} + \cdots + h_{r-1}^{(r)} \frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n} \cdots \sum_{n=0}^{n} (n+1) H_n^{(r)},$$

where

$$h_0^{(r)} + h_1^{(r)} + \cdots + h_{r-1} = 1.$$

To prove this formula, substitute in (4), and then make the substitution

$$(9)\quad \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r)} = \frac{(n+1)\cdots(n+r-1)}{(r-2)!} \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)} - \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-3)!} \cdot \frac{1}{n+1} \sum_{n=0}^{n} H_n^{(r)}$$

$$= (r-1) \frac{(n+1)\cdots(n+r-1)}{(r-1)!} H_n^{(r+1)} - (r-2) \sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!} H_n^{(r+1)},$$

and similarly for other sums. To prove (9), etc., sum by parts once, using the formula

$$\sum_{n=0}^{n} u(n)\Delta v(n) = u(n)v(n)\Big]_{0}^{n+1} - \sum_{n=0}^{n} v(n+1)\Delta u(n).$$

Notice that the coefficients in the last member of (9), namely $(r-1)$ and $-(r-2)$, add to unity, and then by mathematical induction formula (8) is proved.

The conclusions of the theorem are readily drawn from equations (5) and (8). Consider first that $C_n^{(r)} \to s$. Then by a repetition of the arithmetic mean theorem each sum in (5) approaches s and hence

$$H_n^{(r)} \to (k_0^{(r)} + k_1^{(r)} + \cdots + k_{r-1}^{(r)})s = s.$$

Next suppose that $H_n^{(r)} \to s$. We do not have a theorem so well known as the arithmetic mean theorem to refer to but, if we notice that when $H_n^{(r)}$ is replaced by a constant s, each sum in (8), as

$$\frac{r!}{(n+1)\cdots(n+r)} \sum_{n=0}^{n}\sum_{n=0}^{n} \frac{(n+1)\cdots(n+r-2)}{(r-2)!}s,$$

equals s, a little simple epsilon work gives the desired result that $C_n^{(r)} \to s$.

Difference equations from which the coefficients $k_i^{(r)}$ and $h_i^{(r)}$ can be calculated might be written down but are omitted as they are not necessary for the proof of the theorem.

HUNTER COLLEGE

SOME PROPERTIES OF CONTINUOUS CURVES*

BY G. T. WHYBURN

The points A and B of a continuum M are said to be separated in M by a point X of M if $M-X$ is the sum of two mutually separated sets S_1 and S_2 containing A and B respectively. The point P of a continuum M is a cut point of M if and only if the set of points $M-P$ is not connected, i.e., is the sum of two mutually separated point sets.

A continuous curve M will be said to be *cyclicly connected* provided that every two points of M lie together on some simple closed curve which is contained in M. In this paper use will be made of the following fundamental theorem.

THEOREM A. *In order that the continuous curve M should be cyclicly connected it is necessary and sufficient that M should have no cut point.*

A proof for Theorem A will be found in my paper *Cyclicly connected continuous curves*, which will appear soon.

THEOREM I. *If A and B are any two points of a continuous curve M and if K denotes the set of all those points of M which separate A from B in M, then $K+A+B$ is a closed set of points.*

PROOF. The curve M contains a simple continuous arc t from A to B. Clearly K must be a subset of t. Let P be any point of $t-(K+A+B)$. Since P does not belong to K, A and B must both belong to some connected subset† of $M-P$, and by a theorem of R. L. Moore's‡ it follows that $M-P$ contains an arc t' from A to B. On the arcs PA

* Presented to the Society, December 31, 1926.

† See an abstract of a paper by R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, this Bulletin, vol. 32 (1926), pp. 217–218.

‡ *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255.

and PB of t, in the order from P to A and from P to B respectively, let X and Y respectively denote the first points belonging to t'. Then no point of the segment XPY of t can belong to $K+A+B$. Since thus every point of $t-(K+A+B)$ belongs to some segment of t which contains no point of $K+A+B$, the set $K+A+B$ is closed.

DEFINITION. A cyclicly connected continuous curve C is said to be a *maximal cyclic curve* of a continuous curve M if C is a subset of M and is not a proper subset of any other cyclicly connected continuous curve belonging to M.

THEOREM II. *If A and B are any two points of a continuous curve M, t is any arc of M from A to B, K denotes the set of all those points of M which separate A from B in M, and S is any maximal segment of $t-(K+A+B)$, then M contains a maximal cyclic curve which contains S.*

PROOF. Let E and F denote the end points of S. Let G denote the collection of all the maximal connected subsets of $M-(E+F)$. At least one element of G, namely, the one which contains S, must have both of the points E and F for limit points. Only a finite number of elements of G can have this property. Let H denote the point set obtained by adding together all the elements of G which do have this property, and let N denote the point set $H+E+F$. Then N is closed and connected. The continuum N is also connected im kleinen. For since M is connected im kleinen, no point of H is a limit point of $M-H$. Hence H is connected im kleinen. Then from a theorem of R. L. Moore's* it easily follows that $H+E+F=N$ must also be connected im kleinen. Thus we see that N is a continuous curve.

Now if N has no cut point, then by Theorem A, N must be cyclicly connected. If N has any cut points, then for every cut point X of N, let H_x denote the maximal connected subset of $N-X$ which contains $(S+X)-X$, and let N_x denote the point set H_x+X. That H_x exists in case X is

* *A report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), pp. 296–297.

not on S is obvious. If X belongs to S, then since X does not belong to $K+A+B$, $M-X$ contains an arc* t' from A to B. The arc t' must contain the points E and F, for E and F belong to $K+A+B$. The arc EF of t' must belong to N, hence also to $N-X$. Therefore, $S-X$, being the sum of the segments EX and EF of t, must lie in some connected subset of $N-X$. Thus, in any case, H_x exists. Now let L denote the set of all points which are common to all the point sets N_x. Clearly L exists, is a closed point set, and contains S. Let C denote the maximal connected subset of L which contains S. I shall show that C is a maximal cyclic curve of M.

Clearly C is closed and connected. I shall first show that if P and Q are any two points of C and POQ is any arc of N from P to Q, then every point of POQ must belong to C. This must be true, for if X is any cut point of N not on POQ, then H_x contains every point of POQ because it contains P and Q. And if X is any cut point of N on POQ, then H_x contains $POQ-X$, because $POQ-X$ is either a single connected set containing a point of H_x or the sum of two connected sets each of which contains a point of H_x. Hence, in any case, POQ belongs to every set N_x and therefore belongs to L and to C.

The continuum C is a continuous curve. For let P be any point of C and ϵ any positive number. Then since N is connected im kleinen, there exists a positive number δ_ϵ such that every point of N whose distance from P is less than δ_ϵ can be joined in N to P by an arc which is of diameter less than ϵ. Let X be any point of C whose distance from P is less than δ_ϵ. Then N contains an arc a from X to P of diameter less than ϵ. But as was shown above, a must belong to C. Hence C is connected im kleinen at every one of its points and is therefore a continuous curve.

The curve C has no cut point. For suppose, on the contrary, that C has a cut point X. Then $C-X=S_1+S_2$,

* See R. L. Wilder, loc. cit., and R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

where S_1 and S_2 are mutually separated point sets. Let P_1 and P_2 be points of S_1 and S_2 respectively. Now if X is a cut point of N, then since H_x contains P_1 and P_2, it follows* that $N-X$ contains an arc s from P_1 to P_2. And if X is not a cut point of N, again it follows that $N-X$ contains an arc s from P_1 to P_2. And in either case, as was shown above, the arc s must belong to C. Hence S_1 and S_2 are not mutually separated, contrary to supposition. Therefore C has no cut point, and by Theorem A it follows that C is a cyclicly connected continuous curve. That C is a maximal cyclic curve of M follows immediately from the facts (1) that every arc of M joining two points of N must belong wholly to N and (2) that every arc of N joining two points of C must belong wholly to C. This completes the proof.

THEOREM III. *If A and B are any two points of a continuous curve M and if K denotes the set of all those points of M which separate A from B in M, then M contains two simple continuous arcs t_1 and t_2 from A to B whose common part is $K+A+B$.*

PROOF. By Theorem I, $K+A+B$ is a closed set of points. Hence if t is any arc of M from A to B, t must contain K, and $t-(K+A+B)$ is the sum of a countable number of non-overlapping segments $S_1, S_2, S_3, \cdots$. By Theorem II it follows that for every positive integer i, M contains a maximal cyclic curve C_i which contains S_i. For each i, let the end points of S_i be denoted by A_i and B_i. Since C_i is cyclicly connected, then for each i, C_i contains two arcs t_{1i} and t_{2i} from A_i to B_i whose common part is only the points A_i and B_i. Let

$$t_1 = K + A + B + \sum_{i=1,2,3,\cdots} t_{1i}, \qquad t_2 = K + A + B + \sum_{i=1,2,3,\cdots} t_{2i}.$$

Then t_1 and t_2 are simple continuous arcs of M from A to B whose common part is $K+A+B$.

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* See R. L. Moore, loc. cit.

THE DUAL OF A LOGICAL EXPRESSION*

BY B. A. BERNSTEIN

The Peirce-Schröder law of duality in Boolean logic is that for every proposition in the logic there exists another proposition, obtained from the former by interchanging the operations $+$, $\times$ and the elements 0, 1. The rule for obtaining the dual of a logical expression involved in this law is convenient enough when no special form is desired for the dual, but it is generally not at all convenient if the dual is required in the very much desired *normal* form. The main object of this note is to obtain a convenient rule for writing down the dual of an expression in the normal form.

Let $a, b, \cdots, l$ be the *discriminants* of a logical function $f(x, y, \cdots, t)$. Then f, developed normally with respect to its arguments, is given by

$$(1)\quad f(x,y,\cdots,t) = axy\cdots t + bxy\cdots t' + \cdots + lx'y'\cdots t',$$

where the primes indicate negation. If f_1 denote the dual of f, we have by the rule of Peirce and Schröder

$$f_1 = (a_1 + x + y + \cdots + t)(b_1 + x + y + \cdots + t')\cdots (l_1 + x' + y' + \cdots + t'),$$

where $a_1, b_1, \cdots, l_1$ are the respective duals of $a, b, \cdots, l$.† Or, developed normally with respect to $x, y, \cdots, t$,

$$(2)\quad f_1 = l_1xy\cdots t + \cdots + b_1x'y'\cdots t + a_1x'y'\cdots t'.$$

Let us call two discriminants of a function *conjugate* when the arguments associated with one are the respective negatives of those associated with the other. Then (2) tells us

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* Presented to the Society, San Francisco Section, October 30, 1926.

† The discriminants $a, b, \cdots, l$ may be 0, 1, or functions of other elements, $\alpha, \beta, \cdots, \mu$; so that $a_1, b_1, \cdots, l_1$ will in general be different from $a, b, \cdots, l$ respectively.

that *any discriminant of the dual of a function is the dual of the conjugate of the corresponding* discriminant of the function.* We then have the following rule.

RULE. *To obtain the dual of a logical expression, develop the expression normally with respect to any of its elements and change each discriminant to the dual of its conjugate.*

This is the desired rule. Thus, the dual of

$$axy + abx'y + x'y'$$

is

$$0 \cdot xy + (a + b)xy' + x'y + ax'y'.$$

Our rule brings to light some interesting facts about duals. We have at once that *a function is self-dual if each discriminant is the dual of its conjugate.*

Further, the negative of f given by (1) is

$$(3) \quad f' = a'xy \cdots t + b'xy \cdots t' + \cdots + l'x'y' \cdots t'.$$

So that

$$(f')_1 = (l')_1xy \cdots t + \cdots + (b')_1x'y' \cdots t + (a')_1x'y' \cdots t'$$

and

$$(f_1)' = (l_1)'xy \cdots t + \cdots + (b_1)'x'y' \cdots t + (a_1)'x'y' \cdots t'.$$

But

$$(k')_1 = (k_1)' \quad \text{for} \quad k = 0, 1, \alpha + \beta, \alpha\beta.$$

Hence

$$(4) \qquad (f')_1 = (f_1)' = f_1'.$$

That is, *the dual of the negative of a function is the negative of the dual of the function.*†

With the aid of (4), we get

$$(5) \qquad (((f')_1)')_1 = (((f_1)')_1)' = \cdots = f_{11}'' = f.$$

And so, *a function is left unaltered if operated on in any order by duals and negatives each taken an even number of times.*

* By *corresponding* discriminants of two functions we mean discriminants associated with identical products of the arguments.

† Sheffer's operation $a|b$ is seen to be the dual-negative of ab.

Again, generally $f_1 \neq f'$. Under what condition will $f_1 = f'$? From (2) and (3) we see that *the dual of a function is the same as the negative of the function if each discriminant is the dual-negative of its conjugate.*

Finally, if we have a relation $f = 0$, then in general $f_1 \neq 1$, though always $f' = 1$. What is the condition that $f_1 = 1$ when $f = 0$? By means of (2) and (3) we find this condition to be *the same as the condition that* $f_1 = f'$.

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THE HEAVISIDE OPERATIONAL CALCULUS*

BY H. W. MARCH

In a number of recent papers, Carson† has made a definite advance in the study of the Heaviside operational calculus by showing that the solution of an operational equation of the type in question can be obtained from an integral equation. Having done this, he was able to discuss Heaviside's three principal rules and to derive a number of important theorems by the use of which it is possible to solve by operational methods, problems to which Heaviside's rules are not directly applicable.

Somewhat earlier Bromwich‡ and Wagner§ solved, by the use of contour integrals in the complex plane, problems to which one of Heaviside's rules is applicable. They noted that the corresponding rule of Heaviside, the expansion theorem, follows at once from a calculation of the residues at the poles of the integrand in the case of a suitably restricted

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* Presented to the Society, December 31, 1926.

† J. R. Carson, Bell Technical Journal, vol. 4 (1925), pp. 685–761; vol. 5 (1926), pp. 50–95, 336–384; this Bulletin, vol. 22 (1926), pp. 43–68. See also P. Levy, Bulletin des Sciences Mathématiques, vol. 50 (1926), pp. 174–192.

‡ T. J. I'A. Bromwich, Proceedings of the London Society, (2), vol. 15 (1916), pp. 401–448; see also H. S. Carslaw, Philosophical Magazine, vol. 39 (1920), pp. 603–611.

§ K. W. Wagner, Archiv für Elektrotechnik, vol. 4 (1916), pp. 159–193.

operator. In a later paper Bromwich* also obtained from a contour integral the asymptotic solution for a particular problem.

It is the purpose of the present paper to show the connection between the methods of Carson and Bromwich by showing that Bromwich's contour integral is the solution of the integral equation set up by Carson. In the opinion of the writer the two methods supplement one another in a valuable way. To emphasize the importance of the contour integral in obtaining Heaviside's rules rigorously and simply, the derivation of each of the three principal rules is briefly indicated.

Carson showed that the solution $h(t)$ of the operational equation†

$$h(t) = \frac{1}{H(p)} \tag{1}$$

which satisfies the appropriate initial conditions is the solution of the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt, \tag{2}$$

the real part of p being assumed to be negative.

Bromwich, on the other hand, attacked the problem directly without using the operational calculus, by assuming the solution of the corresponding differential equation or system of differential equations to be expressed by a contour integral in the complex plane. He found that the solution $h(t)$ is given in certain cases by the integral‡

$$h(t) = \frac{1}{2\pi i} \int_C \frac{e^{pt}}{pH(p)}dp, \tag{3}$$

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* Proceedings, Cambridge Philosophical Society, vol. 20 (1920–21), pp. 423–427.

† For notation and details, see Carson, Bell Technical Journal, vol. 4 (1925), pp. 686–705; this Bulletin, vol. 22 (1926), pp. 43–47.

‡ Proceedings of the London Society, (2), vol. 15 (1916), pp. 410–420.

or by the integral

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt\infty}}{pH(p)} dp, \tag{4}$$

where C in (3) is a closed curve surrounding all the poles of the function

$$\frac{1}{pH(p)},$$

no one of which has, by hypothesis, a positive real part, and where the path of integration in (4) is a straight line parallel to the axis of imaginaries, c being any positive real number. The path of integration in (3) results from a deformation of the path in (4).

It will now be shown that the solution* of the integral equation (2) is given by (4). Fourier's theorem can be written in the form†

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xy} dy \int_0^\infty e^{-yz} f(z) dz, \tag{5}$$

where c and the path of integration are chosen as in (4). If we write

$$g(y) = \int_0^\infty e^{-zy} f(z) dz, \tag{6}$$

it follows from (5) that

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(y) e^{xy} dy. \tag{7}$$

Accordingly (7) furnishes the solution‡ of the integral equation (6) provided that $g(y)$ is subject to suitable restrictions. These must be such that (6) can be shown to be satisfied when $f(z)$ as given by (7) is substituted in it. This

* See the recent paper by J. D. Tamarkin, Transactions of this Society, vol. 28 (1926), p. 417.

† See, e.g., H. M. MacDonald, Proceedings of the London Society, vol. 35 (1902), p. 428. See also B. Riemann, *Gesammelte Werke*, p. 149.

‡ For a proof of the uniqueness of a continuous solution of equation (6), see M. Lerch, Acta Mathematica, vol. 27 (1903), pp. 339–351.

can be done if $g(y)$ is an analytic function of y at all points in the half-plane in which the real part of y is positive and if further, writing $y=\rho e^{i\theta}$, it is true that in this half-plane

$$|g(y)| < \frac{N}{\rho^{\alpha}},$$

for a sufficiently large ρ, N being a sufficiently large fixed positive number, and α being a fixed positive number. In accordance with (7) the function $h(t)$ defined by equation (4) in terms of Bromwich's contour integral is the solution of Carson's integral equation (2).

Heaviside's rules can be derived in a much more direct and rigorous manner from the contour integral than from the integral equation. Indeed the contour integral appears to furnish the key to the whole situation, making it possible to determine whether or not in a given case the Heaviside rule in question is applicable and to discuss in a satisfactory manner the results obtained by applying these rules. If it appears in a given case that Heaviside's rules are not applicable, the result is to be sought by studying the contour integral itself.

We shall now indicate briefly the derivation of Heaviside's three principal rules and give an example in which it can be seen from the contour integral why Heaviside's rule does not lead to a correct result.

(a) *The Power Series Solution.* Assume that the function $H(p)$ is such that the path of integration in (4) can be deformed into a circle C described about the origin as center, and that on C, $1/H(p)$ can be expanded in the convergent series

$$\frac{1}{H(p)} = a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n} + \cdots.$$

Also expand e^{pt} in a power series in p. Then

$$h(t) = \frac{1}{2\pi i}\int_C \left(\frac{a_0}{p} + \frac{a_1}{p^2} + \frac{a_2}{p^3}\cdots\right) \cdot\left(1 + pt + \frac{p^2t^2}{2!} + \frac{p^3t^3}{3!} + \cdots\right)dp.$$

On forming the product of these series term by term, and arranging according to negative and positive powers of p, we find the Laurent expansion of the integrand in a circular ring enclosing the origin as center. The value of the integral is $2\pi i$ times the coefficient of the term in $1/p$. Hence we have

$$h(t) = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \cdots + \frac{a_n t^n}{n!} + \cdots.$$

(b) *The Expansion Theorem.** Assume that $H(p)$ is a rational function which vanishes for n distinct values of p, but which does not vanish for $p=0$, and that the path of integration can be deformed into a circle c described about the origin as center and enclosing all the roots of $H(p)=0$. On calculating the residues at the points $0, p_1, p_2, \cdots, p_n$, the poles of the integrand, it is found that

$$h(t) = \frac{1}{H(0)} + \sum_{j=1}^{j=n} \frac{e^{p_j t}}{p_j H'(p_j)}.$$

The same treatment applies when $H(p)=0$ has multiple roots or when $p=0$ is a root. By an appropriate deformation of the path of integration it is possible to extend the method to cases in which $H(p)$ is a transcendental function.

(c) *The Asymptotic Solution.* The contour integral lends itself equally well to the discussion of the third of Heaviside's principal rules, the asymptotic solution, and the one for which Carson's treatment was not entirely satisfactory. The method given below of obtaining the asymptotic solution was developed independently by the writer but its essential features were discovered earlier by Bromwich† in treating a particular case.

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* The proof outlined here was published almost simultaneously by Bromwich and Wagner in 1916. For other interesting derivations of the expansion theorem, see Bromwich, Philosophical Magazine, vol. 37 (1919), pp. 407–19; and F. D. Murnaghan, this Bulletin, vol. 33 (1927), p. 81.

† Proceedings, Cambridge Philosophical Society, vol. 20 (1920–21), pp. 423–427.

Heaviside's rule states that if $1/H(p)$ can be expanded in the form

$$(8)\quad \frac{1}{H(p)} = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots + (b_0 + b_1 p + b_2 p^2 + \cdots + b_n p^n + \cdots) p^{1/2},$$

$h(t)$ is obtained by discarding all terms in the first line excepting a_0, by replacing $p^{1/2}$ in the second line by $1/(\pi t)^{1/2}$ and p^n in this line by d^n/dt^n. It results that

$$(9)\quad h(t) = a_0 + \left(b_0 + b_1\frac{d}{dt} + b_2\frac{d^2}{dt^2} + \cdots\right)\frac{1}{(\pi t)^{1/2}}$$
$$= a_0 + \frac{1}{(\pi t)^{1/2}}\left(b_0 - b_1\frac{1}{2t} + b_2\frac{1\cdot 3}{(2t)^2} - b_3\frac{1\cdot 3\cdot 5}{(2t)^3} + \cdots\right).$$

Assume that $H(p)$ is such a function that the path of integration of (4) can be deformed into the path $ANPQB$ (see Fig. 1) enclosing the origin. This can be done in all

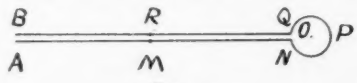

FIG. 1

cases which I have examined in which the formula can be applied and presumably in all such cases.

If $1/H(p)$ can be expanded in the form (8), we note that the integral of each term of the expression

$$\frac{1}{p}(a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots)e^{pt}$$

along the path $ANPQB$ is zero, while

$$\frac{1}{2\pi i}\int \frac{a_0}{p} e^{pt} dp = a_0,$$

the integral being taken along the same path as before. Accordingly, if we pass over, for the moment, all questions of convergence and make use of a proper determination

of $p^{1/2}$ by writing $p^{1/2} = -i\rho^{1/2}$ along AN and $p^{1/2} = i\rho^{1/2}$ along QB, we find

$$
\begin{aligned}
h(t) &= a_0 + \frac{1}{2\pi i}\int_{AN} \frac{e^{pt}}{p^{1/2}}(b_0 + b_1 p + b_2 p^2 + \cdots)dp \\
&\quad + \frac{1}{2\pi i}\int_{QB} \frac{e^{pt}}{p^{1/2}}(b_0 + b_1 p + b_2 p^2 + \cdots)dp \\
&= a_0 + \frac{1}{\pi}\int_0^\infty (b_0 - b_1\rho + b_2\rho^2 - b_3\rho^3 + \cdots)\frac{e^{-\rho t}}{\rho^{1/2}}d\rho
\end{aligned} \tag{10}
$$

Equation (9) follows immediately on integrating term by term.

This analysis, which is purely formal, has led to the asymptotic solution and gives an intelligible foundation for this rather strange rule of Heaviside's. In certain cases the series in (9) is convergent for all positive values of t, in other cases it is only asymptotically convergent, and in others it is meaningless.* In any case, it is necessary to study the integral and the calculation outlined in obtaining equations (9) and (10). If the coefficients $b_0, b_1, b_2 \cdots$ are such that the series (9) is convergent for positive values of t, the investigation is usually not difficult. If the series is only asymptotically convergent, it will be found in many cases that we can proceed in the following manner. If possible, write the operator in the form

$$
\begin{aligned}
\frac{1}{pH(p)} &= \frac{1}{p}(a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n) \\
&\quad + (b_0 + b_1 p + b_2 p^2 + \cdots + b_n p^n + r_{n+1}p^{n+1})p^{1/2},
\end{aligned}
$$

in which corresponding to a fixed n,

$$
|r_{n+1}| < A,
$$

provided that $|p| < k$, A being a fixed positive number. Let the points corresponding to $p = -k$ be denoted by M and R in Fig. 1. It can usually be shown that t can be chosen so large that the integrals of $e^{pt}/(pH(p))$ from $-\infty$ to M and

* Carson, Bell Technical Journal, vol. 4 (1925), pp. 744 and 749.

from R to $-\infty$, are arbitrarily small in absolute value. Also t can be chosen so large that the integrals from M to N and from Q to R of $r_{n+1}\, p^{n+1/2}\, e^{pt}$ is arbitrarily small. Then for a sufficiently large t, the absolute value of the difference between $h(t)$ and

$$a_0 + \frac{1}{\pi} \int_0^k [b_0 - b_1\rho + b_2\rho^2 - \cdots + b_n(-\rho)^n] \frac{e^{-\rho t}}{\rho^{1/2}} d\rho$$

is an arbitrarily small positive number. After this, the limit k can be replaced by ∞ for a sufficiently large t. Then $h(t)$ is given approximately by

$$h(t) = a_0 + \frac{1}{(\pi t)^{1/2}} \left[b_0 - \frac{b_1}{2t} + b^2 \frac{1 \cdot 3}{(2t)^2} - b_3 \frac{1 \cdot 3 \cdot 5}{(2t)^3} + \cdots + (-1)^n b_n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2t)^n} \right].$$

The importance of the contour integral in determining the applicability of Heaviside's rules is illustrated in the following example. Carson* showed that the application of the method of the asymptotic solution to the operational equation

$$h(t) = \frac{p^{1/2}}{p^2 + \omega^2},$$

led to an incorrect result. The reason that it does so is readily seen from a consideration of the contour integral

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p^{1/2}}{p^2 + \omega^2} e^{pt} dp.$$

The path of integration cannot be deformed into the path of Fig. 1, because of the presence of singularities of the integrand at the points $p = \pm i\omega$. It is accordingly not to be expected that the asymptotic solution can be employed.

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* Bell Technical Journal, vol. 4 (1925), p. 749.

EXTENSIONS OF WARING'S THEOREM ON FOURTH POWERS*

BY L. E. DICKSON

1. *Introduction.* In 1770 Waring conjectured that every positive integer p is a sum of nineteen integral biquadrates. It is shown in § 8 that eight of them may be taken equal if $p \leqq 4100$. Again, sixteen of them may be taken equal in pairs if $p \leqq 2400$. All possible similar results are included in Theorem 1 of § 3.

2. *Notations and Definitions.* The form

$$(1) \qquad (a_1, \cdots, a_n) = a_1x_1^4 + \cdots + a_nx_n^4, \qquad (0 < a_1 \leqq a_2 \leqq a_3 \cdots),$$

is said to be of *order* n and *weight* $a_1+a_2+\cdots+a_n$. Since $ax^4 = x^4+\cdots+x^4$, to a terms, a form of weight w is equal to a sum of w biquadrates. But 79 is not a sum of fewer than nineteen biquadrates. Hence 19 is the minimum weight of a form (1) which represents all positive integers.

Let f be a form (1) which represents p, and let $a_1 = r+s$. The form $g = (r, s, a_2, \cdots, a_n)$ shall be said to be derived from f by the *partition* of a_1 into $r+s$. If we give to the first two variables in g the same value x_1 as was employed in $f = p$, we see that also g represents p. Hence *any form derived from f by partition represents every integer which can be represented by f* (and usually represents further integers).

Write $a = 2^4$, $b = 3^4$, $c = 4^4$, $\cdots$. If a positive integer m can be expressed as a linear combination of $1, a, b, \cdots$, with integral coefficients $\geqq 0$ whose sum is $\leqq 19$, in one and only one way, m shall be called a *simple* number. In case there are exactly two such expressions, m shall be called a *double* number. Similarly for a *triple* or *k-fold* number.

* Presented to the Society, December 31, 1926.

For example, $19=3+a$ is a double number, while $4+a$ is a simple number.

We write 1_2 for 1, 1; and 1_s for s ones.

3. *Summary of Results.*

THEOREM 1. *If a form of weight* 19 *represents all positive integers, it is* $A=(1122337)$, $B=(1122346)$, $C=(11112238)$, $D=(11112247)$, *or* $E=(11122228)$, *or a form derived from one of the five by partition.*

It is proved* in §§ 6, 7 that both A and B (of the minimum order 7) represent every positive integer $p \leqq 7^4=2401$. A like method was used to verify that D represents every $p \leqq 1300$, and that $F=(1122238)$ represents every $p \leqq 2000$ except the four-fold number

$$443 = 9 + a + 2b + c = 10 + 6a + b + c$$
$$= 6 + 2a + 5b = 7 + 7a + 4b.$$

The partitions $2=1+1$ and $3=1+2$ of F give C and E, respectively, and they represent $1+8+a+2b+d$. Hence C and E both represent every $p \leqq 2000$.

A form is evidently more likely to represent the integers just exceeding a given biquadrate than those just preceding it. As an excellent further check, it was verified that A, B, and F each represent 4000 up to $4096=8^4$. The same is therefore true of C and E.

4. *Conditions that a Form of Weight* 19 *shall Represent all Integers.* Let such a form f have the notation (1). Then

$$a_1 + a_2 + \cdots + a_n = 19. \tag{2}$$

To prove that

$$a_k \leqq 1 + a_1 + a_2 + \cdots + a_{k-1} \qquad (k = 2, \cdots, n), \tag{3}$$

write s for $a_1+\cdots+a_{k-1}$ and suppose that $a_k>1+s$. For $i=1, \cdots, k-1$, partition a_i into as many ones. Thus a

* Mr. K. C. Yang kindly extended my computations from 2000 to 2400 for A and B, and from 1000 to 1300 for D and E.

partition of f is $(1_s, a_k, \cdots, a_n)$, which does not represent $1+s$. The latter is a simple number since $19 \geqq s+a_k > 1+2s$, whence $1+s<10$.

Since f represents 1, $a_1=1$. By (3) with $k=2$, $a_2 \leqq 2$. If $a_2=2$, f would not represent the simple number $110=13+a+b$. Apply (3) with $k=3$. Hence

$$a_1 = 1, \quad a_2 = 1, \quad a_3 \leqq 3. \tag{4}$$

If $a_n \geqq 9$, a partition of f is $(1_{10}, 9)$ which does not represent the simple number $8+3a$. Hence

$$a_n \leqq 8. \tag{5}$$

By (2), (4), (5), we see that $n>4$.

If $a_{n-1}>4$, a partition of f is $(1_9, 5, 5)$, which does not represent the simple number $235=9+4a+2b$, since 9 is not a sum of certain of (1, 1, 1, 5, 5). Hence

$$a_{n-1} \leqq 4. \tag{6}$$

If $a_{n-2}>3$, a partition of f is $G=(1_7, 4, 4, 4)$. But G does not represent the simple number $221=11+3a+2b$, since 11 is not a sum of certain of (1, 1, 4, 4, 4). Hence

$$a_{n-2} \leqq 3. \tag{7}$$

Let $(111a_4 \cdots)$ with $a_4>2$ represent the simple number $15+2a$. Then 15 is a sum of certain of $1, a_4, \cdots, a_n$, contrary to $a_4+\cdots+a_n=19-3=16$, $a_i \geqq 3$.

Let $(113\cdots)$ represent $15+2a$. Then 15 is a sum of certain of $(3, a_4, \cdots, a_n)$, contrary to $a_4+\cdots+a_n=14$, $a_i \geqq 3$.

These two results and (4) give

(8) $a_3=1$ or 2. If $a_3=1$, then $a_4=1$ or 2.

The following forms are excluded:

$$(1_5,3,3,3,5), \quad (1_5,3,3,4,4), \quad (1_4,3_5), \quad (1_2,2_6,5). \tag{9}$$

No one of the first three represents the simple number $207=13+2a+2b$. The fourth does not represent the simple number $141=12+3a+b$.

A partition of $(1_7, 4, 8)$ is G above (7). Partitions of $(1_5, 3, 3, 8)$ and $(1_5, 3, 4, 7)$ give the first and second forms (9) respectively. This proves

(10) f does not end with 4, 8, or 3, 3, 8, or 3, 4, 7.

Let $a_{n-3}>2$. By (7), $a_{n-3}=a_{n-2}=3$. By (6), $a_{n-1}=3$ or 4. In the second case, a partition gives (9_2). If, in the first case, $a_n \geqq 5$, a partition gives (9_1). This proves

(11) If $a_{n-3}>2$, then $a_{n-3}=a_{n-2}=a_{n-1}=3$, $a_n=3$ or 4.

5. *Proof of Theorem* 1. If $n=5$, (4)–(6) give

$$a_1+\cdots+a_5 \leqq 1+1+3+4+8=17,$$

contrary to (2). Next, let $n=6$. If $a_3=1$, (4)-(7) give

$$a_1+\cdots+a_6 \leqq 1+1+1+3+4+8=18,$$

contrary to (2). Hence $a_3=2$. By (2), (5)-(7),

$$19-4=a_4+a_5+a_6, \quad a_4 \leqq 3, \quad a_5 \leqq 4, \quad a_6 \leqq 8,$$

whence the three signs are all $=$. By (10), the ending 4, 8 is excluded. Hence $n \geqq 7$.

Let $n=7$. By (5)-(7), $a_5 \leqq 3$, $a_6 \leqq 4$, $a_7 \leqq 8$. If $a_3=a_4=1$, (2) gives $f=(1111348)$, which is excluded by (10). Next, let $a_3=1$, $a_4=2$. Then $a_5+a_6+a_7=19-5=14$, whence the sets of values of a_5, a_6, a_7 are 2, 4, 8; 3, 3, 8; 3, 4, 7, all excluded by (10). By (8), there remains only the case $a_3=2$. If $a_4>2$, (11) contradicts (2). Hence $a_4=2$, $a_5+a_6+a_7=13$. If $a_5=3$, f is A or B. If $a_5=2$, f is either (1122247), a partition of which gives the fourth excluded form (9), or else (1122238), the least number not represented by which is the quadruple number

$$443=9+a+2b+c=10+6a+b+c$$
$$=6+2a+5b=7+7a+4b.$$

Let $n=8$. By (5)-(7), $a_6 \leqq 3$, $a_7 \leqq 4$, $a_8 \leqq 8$.

First, let $a_3=1$. Then $a_4 \leqq 2$ by (8). If $a_5>2$, (11) gives $a_5+a_6+a_7+a_8=12$ or 13, while $a_1+a_2+a_3+a_4 \leqq 5$, contrary to the relation (2). Hence we find $a_5 \leqq 2$. If $a_4=2$, then

$a_5=2$, $a_6+a_7+a_8=19-7=12$. The sets of values of a_6, a_7, a_8 are 2, 2, 8; 2, 3, 7; 2, 4, 6; 3, 3, 6; 3, 4, 5. The first gives E. For the second, f is derived from A by the partition $3=1+2$. For the last three sets, f is a partition of B. Finally, let $a_4=1$. If $a_5=2$, the sets of values of a_6, a_7, a_8 are 2, 3, 8; 2, 4, 7; 3, 3, 7; 3, 4, 6. The first two give C and D. The last two yield partitions $2=1+1$ of A and B. If $a_5=1$, the sets are 2, 4, 8; 3, 3, 8; 3, 4, 7, which are excluded by (10).

Second, let $a_3=2$. If $a_4\geqq 3$, f is evidently $(1, 1, 2, 3_5)$, a partition of which is the third excluded form (9). Hence $a_4=2$. If $a_5>2$, f is evidently the partition (11223334) of A. There remains the case $f=(11222a_6a_7a_8)$. The sets of values of a_6, a_7, a_8 are 2, 2, 7; 2, 3, 6; 2, 4, 5; 3, 3, 5; 3, 4, 4. For the first and third sets, the partitions $7=2+5$ and $4=2+2$ respectively yield the fourth excluded form (9). For the second and last sets, f is a partition of B. For the fourth set, f is a partition of A.

Let $n=9$. By (5)-(7), $a_7\leqq 3$, $a_8\leqq 4$, $a_9=8$.

First, let $a_6>2$. By (11), $a_6=a_7=a_8=3$, $a_9=3$ or 4. We get the third excluded form (9) and the partitions $(1_4, 2, 3_3, 4)$ and $(1_3, 2_2, 3_4)$ of A.

Second, let $a_6=1$. Since the ending 4, 8 is excluded, f is evidently one of the partitions $(1_6, 2, 3, 8)$, $(1_6, 2, 4, 7)$, $(1_6, 3, 3, 7)$, $(1_6, 3, 4, 6)$ of C, D, A, B, respectively.

Finally, let $a_6=2$. If $a_5=1$, the symbol for f contains 1_5 and 2, followed by 2, 2, 8; 2, 3, 7; 2, 4, 6; 3, 3, 6; or 3, 4, 5. Thus f is a partition of C, A, B, A, or B, respectively. For $a_5=2$, $a_4=1$, f contains 1_4 and 2_2, followed by 2, 2, 7; 2, 3, 6; 2, 4, 5; 3, 3, 5; or 3, 4, 4, and is a partition of A or B. For $a_5=a_4=2$, $a_3=1$, f contains 1_3 and 2_3, followed by 2, 2, 6; 2, 3, 5; 2, 4, 4; or 3, 3, 4, and is a partition of B. There remains only the case in which f contains 1_2 and 2_4, followed by 2, 2, 5; 2, 3, 4; or 3, 3, 3. The first yields the fourth excluded form (9). The last two yield partitions of B.

Let $n\geqq 10$. The forms which satisfy (2), (5)-(7), and $a_1=a_2=1$ are all partitions of A, B, C or D. We subdivide cases according to the number of 1's in the symbol. For

example if $n=10$, the first case yields only $(1_8, 3, 8)$ and $(1_8, 4, 7)$.

6. *Forms related to B.* The tables employed extend to $7^4=2401$. To the sums of two biquadrates we add the double of each such sum, and obtain a table α of all integers <2401 which are represented by (1122). To each number in α we add the triples of biquadrates and obtain a table β of all integers <2401 which are represented by (11223). To each number in β we add the quadruples of biquadrates and obtain the following table γ of all integers <2401 which are represented by (112234):

0–13, 16–28, 32–43, 48–58, 64–73, 80–93, 96–106, 108, 112–23, 128–38, 144–53, 160–73, 176–88, 192–203, 208–18, 225–33, 241–53, 256–68, 272–83, 288–98, 304–13, 320–33, 336–48, 352–63, 368–78, 384–93, 400–13, 416–28, 432–43, 448–58, 465–73, 481–93, 498–508, 512–23, 528–38, 544–53, 560–73, 576–88, 592–603, 608–18, 624–37, 639–53, 656–68, 672–83, 688–98, 705–15, 717, 721–33, 737–48, 753–63, 768–78, 784–97, 800–13, 816–28, 832–43, 848–58, 864–77, 880–93, 896–908, 912–23, 928–38, 945–57, 961–70, 972–3, 977–88, 993–1003, 1009–18, 1024–37, 1040–50, 1052–3, 1056–68, 1072–83, 1088–98, 1104–17, 1120–30, 1132, 1136–48, 1152–63, 1168–78, 1185–97, 1201–10, 1212, 1217–28, 1233–43, 1249–61, 1265–77, 1280–92, 1296–1308, 1312–23, 1328–41, 1344–57, 1360–72, 1376–88, 1392–1403, 1408–21, 1424–33, 1435–7, 1440–8, 1450–52, 1456–68, 1472–83, 1488–1501, 1505–17, 1521–32, 1536–48, 1552–63, 1568–81, 1584–97, 1600–12, 1616–28, 1632–43, 1648–61, 1664–76, 1680–88, 1690–1, 1696–1708, 1713–23, 1728–41, 1745–56, 1761–72, 1777–88, 1792–1803, 1808–21, 1824–36, 1840–51, 1856–68, 1872–85, 1888–1901, 1904–16, 1920–32, 1934, 1936–48, 1952–65, 1968–81, 1985–96, 1998, 2000–11, 2017–28, 2033–45, 2048–61, 2064–76, 2080–92, 2096–2106, 2108, 2112–25, 2128–41, 2144–56, 2160–72, 2176–88, 2192–2205, 2208–20, 2225–36, 2241–52, 2257–66, 2268, 2273–85, 2290–2300, 2304–16, 2320–32, 2336–46, 2352–65, 2368–80, 2384–95, 2400.

THEOREM 2. *$B_m=(112234m)$ represents all positive integers $\leqq 2400$ if and only if $m=6, 7, 8, 9$.*

The numbers in table γ together with the numbers obtained by adding 6 to them give all numbers $\leqq 2400$ except $p=240$, 480, and $q=1455$. Each $p-6\cdot 2^4$ and $q-6\cdot 3^4$ is in γ. This proves Theorem 2 when $m=6$.

For $m=7$, 8, or 9, the numbers in γ together with the numbers obtained by adding m to them give all numbers $\leqq 2400$.

The least positive integer not represented by B_m is 234, 74, 59, 44, 29, 14 when $m=10, 11, 12, 13, 14, m>14$, respectively.

7. *Forms related to A.* To each number in table β(§ 6) we add the triples of biquadrates and obtain the following table δ of all integers <2401 which are represented by (112233):

0–12, 16–27, 32–42, 48–57, 64–72, 80–92, 96–105, 112–22, 128–37, 144–52, 160–72, 176–87, 192–202, 209–17, 225–32, 241–52, 256–67, 272, 274–82, 288–97, 304–12, 320–32, 336–47, 352–62, 368–77, 384–92, 400–2, 404–12, 416–27, 432–42, 449–57, 465–72, 482, 484–92, 497–507, 512–22, 528–37, 544–52, 560–72, 576–87, 592–602, 608–12, 614–7, 624–36, 640–52, 656–67, 672–82, 689–97, 705–14, 716, 721–7, 729–32, 737–47, 753–62, 768–77, 784–96, 800–12, 816–27, 832–42, 848–57, 864–76, 880–9, 891–2, 896–907, 912–22, 929–37, 945–56, 961–8, 971–2, 977–87, 993–1002, 1009–17, 1024–36, 1040–9, 1051, 1056–63, 1065–7, 1072–82, 1088–97, 1104–16, 1120–9, 1131, 1136–47, 1152–62, 1169–77, 1185–96, 1201–8, 1211, 1217–24, 1226, 1233–7, 1240–2, 1249–60, 1265–76, 1280–91, 1296–1307, 1312–22, 1328–40, 1344–56, 1360–7, 1369–71, 1376–87, 1392–1401, 1408–20, 1424–32, 1434–6, 1440–8, 1450–1, 1456–67, 1472–82, 1489–1500, 1505–16, 1521–31, 1536–47, 1552–62, 1568–80, 1584–93, 1595, 1600–7, 1609–10, 1616–27, 1632–42, 1648–60, 1664–5, 1667–72, 1674–5, 1680–7, 1690–1, 1697–1707, 1712–22, 1729–40, 1745–7, 1749–55, 1761–70, 1778–87, 1792–1802,

1808–20, 1824–7, 1829–33, 1835, 1840–7, 1849–50, 1856–67, 1872–84, 1888–1900, 1904–15, 1920–31, 1934, 1936–47, 1952–64, 1969–77, 1979–80, 1982, 1985–95, 2001–10, 2017–25, 2027, 2033–44, 2048–60, 2064–75, 2080–91, 2096–2107, 2112–24, 2128–39, 2144–55, 2160–71, 2176–7, 2179–85, 2187, 2192–2204, 2209–19, 2225–35, 2241–51, 2257–65, 2274–84, 2290–9, 2304–14, 2320–31, 2336–45, 2352, 2354–64, 2368–79, 2384–5, 2387–94, 2400.

We may now readily verify that $A=(1122337)$ represents all positive integers <2401. The numbers in δ together with the numbers obtained by adding 7 to them give all integers <2401 except

$p = 240,\ 403,\ 480\text{–}1,\ 483,\ 613,\ 976,\ 1216,\ 1232,$
$1238\text{–}9,\ 1673,\ 1696,\ 1748,\ 1828,\ 2273,\ 2353;$
$q = 620,\ 735,\ 1071,\ 1245\text{–}6,\ 1375,\ 1615,\ 1695,\ 1855.$

Each $p-7\cdot 2^4$ and $q-7\cdot 3^4$ is in δ.

Finally, $(112233m)$ fails to represent 621, 622, 73, 58, 43, 13 when $m=8, 9, 10, 11, 12, m\geqq 13$, respectively.

THEOREM 3. *The form* $(112233m)$ *represents all positive integers* <2401 *if and only if* $m=7$.

8. THEOREM 4. *Every positive integer* $\leqq 4100$ *can be expressed in the form* $S+8x^4$, *where* S *is a sum of eleven integral biquadrates, and* $x=0, 1,$ *or* 2.

At the suggestion of Jacobi, Bretschneider* gave a Table 1 of all those decompositions of $1, \cdots, 4100$ into biquadrates >0 for which the number of the latter is a minimum. From it† we conclude that all numbers $\leqq 4100$ are sums of eleven biquadrates $\geqq 0$, except

(12) 12–15, 27–31, 42–47, 57–63, $\cdots$, 4091–5.

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* Journal für Mathematik, vol. 46 (1853), pp. 1–23.

† Or by taking the aggregate of the numbers in Parts XI–XVIII of his Table 2, which lists the integers which are sums of a prescribed number of biquadrates >0.

If p occurs in the set (12), $p-8$ is not in it except for $p=232, 240, 472, 480$. For these four, $p-8\cdot 2^4$ is not in the set (12). This proves Theorem 4.

9. Corresponding results for cubes are obtained in the writer's paper in the American Mathematical Monthly for April, 1927. Assistance has been provided by the Carnegie Institution for the more elaborate investigation of fifth and higher powers.

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TESTS FOR PRIMALITY BY THE CONVERSE OF FERMAT'S THEOREM*

BY D. H. LEHMER

There are, generally speaking, two distinct methods for determining the primality of a large integer without trying possible divisors. Up to this time the method which goes by the name of Lucas' test† has yielded the most results. It is particularly well adapted to the investigation of Mersenne numbers and has consequently led to the identification of the three largest primes heretofore known, namely, $2^{89}-1$, $2^{107}-1$ and $2^{127}-1$. The other method is based on the converse of Fermat's theorem. It is the purpose of this paper to discuss certain improvements in this method, and to apply it to some numbers of the form $10^n \pm 1$.

It has long been known that the simple converse of Fermat's theorem, namely: *If* $a^x \equiv 1 \pmod N$ *for* $x=N-1$, *then* N *is a prime*, is *not* true, as is shown by the simple example: $4^{14} \equiv 1 \pmod{15}$. A true converse of this theorem was first given by Lucas‡ in 1876: *If* $a^x \equiv 1 \pmod N$ *for* $x=N-1$, *but not for* $x<N-1$, *then* N *is a prime.* In 1891 he proved the following theorem.§

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* Presented to the Society, San Francisco Section, April 2, 1927.

† American Journal, vol. 1 (1878), pp. 184–220.

‡ Lucas, loc. cit., p. 302.

§ *Théorie des Nombres*, 1891, pp. 423, 441.

THEOREM 1. *If* $a^x \equiv 1 \pmod{N}$ *for* $x = N-1$, *but not for* x *a proper divisor of* $N-1$, *then* N *is a prime.*

When applied to a particular N, this theorem exhibits three defects. In the first place, the complete factorization of $N-1$ must be known. Secondly, the number of values of x which must be tried in order to show that the second part of the hypothesis is fulfilled, may be impossibly large. Thirdly, the condition for primality is sufficient but not necessary. If, however, N is of the form 2^n+1, the first two defects vanish; for in this case all the divisors of $N-1$ are powers of 2, so that in testing for the first part of the hypothesis, the second part is automatically taken care of in the successive squarings of the residues modulo N. Unfortunately the only numbers of the form 2^n+1 that have any chance to be primes are the Fermat numbers in which n is a power of 2. The numbers $2^{128}+1$ and $2^{256}+1$ have been tested in this way by Morehead and A. E. Western,* and both numbers were found to be composite. The next such number awaiting investigation is $2^{1024}+1$, a number of 309 digits. A skillful computer could test this number in about ten years. As far as is known to the present author, no prime above the range of ordinary methods of factorization has ever been identified by the converse of Fermat's theorem.† It is clear that Theorem 1 must be improved before further results are possible.

The first defect has to do with the factorization of $N-1$, and is difficult to overcome in case N is a number of unknown form. In many cases it is no easier to factor $N-1$ completely than to factor N. If, however, N is the maximum algebraically prime factor of a number of the form $y^n \pm 1$, it is usually

* This Bulletin, vol. 11 (1905), pp. 543; and vol. 16 (1909), pp. 1–6.

† An attempt to establish the primality of $2^{61}-1$ was made by Seelhoff (Zeitschrift für Mathematik und Physik, vol. 31 (1886), pp. 174–178) using methods depending on the converse of Fermat's theorem. The proof is invalid, as was pointed out by Cole. It depends upon the false proposition that if a divides b and also c, then either b divides c or else c divides b. Despite this obvious error, the proof of the primality of $2^{61}-1$ is invariably attributed to Seelhoff.

possible to decompose $N-1$ into algebraic factors as is shown in the following examples:

$$\frac{y^n \pm 1}{y \pm 1} - 1 = \frac{y}{y \pm 1}(y^{n-1} - 1),$$

$$\frac{y^{4n+2} \pm 1}{y^2 \pm 1} - 1 = \frac{y^2}{y^2 \pm 1}(y^{4n} - 1),$$

$$\frac{y^{4n} \pm 1}{y^4 \pm 1} - 1 = \frac{y^4}{y^4 \pm 1}(y^{4n-4} - 1),$$

$$\frac{y^{8n} \pm 1}{y^8 \pm 1} - 1 = \frac{y^8}{y^8 \pm 1}(y^{8n-8} - 1),$$

$$\frac{y^{n^2} \pm 1}{y^n \pm 1} - 1 = \frac{y^n}{y^n \pm 1}(y^{n^2-n} - 1),$$

. . . . . .

The large factors on the right are differences of squares. Thus the factors of $N-1$ are made to depend upon factorizations of $y^n \pm 1$ for much smaller values of n. These may be taken from tables* previously computed. In case $N-1$ cannot be expressed in some such form as the above, one can always be assured of having n as a factor of $N-1$. If a small factor of N is known (as it often is from congruence tables) and if N' is the residuary factor, the factorization of $N'-1$ is again very difficult. In this case, however, we have n as a factor, but it is of little use except in reducing the size of the number to be factored. We shall see later that it is not necessary to know the complete factorization of $N-1$.

The next improvement in Theorem 1 that suggests itself is the reduction of the number of values of x to be tried. In the following theorem this number is reduced from the number of divisors of $N-1$ to the number of its prime factors.

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* Cunningham and Woodall, *Factorization of* $y^n \pm 1$, London, 1925.

THEOREM 2. *If $a^x \equiv 1 \pmod{N}$ for $x = N-1$, but not for x a quotient of $N-1$ on division by any of its prime factors, then N is a prime.*

Let

$$N - 1 = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t} = m_i p_i, \qquad (i = 1,2,3, \cdots, t).$$

Then we have for an hypothesis:

$$a^{N-1} \equiv 1 \pmod{N} \tag{1}$$

$$a^{m_i} \not\equiv 1 \pmod{N}, \qquad (i = 1,2,3, \cdots, t). \tag{2}$$

Now let ϵ be the smallest value of x satisfying $a^x \equiv 1 \pmod{N}$. Then ϵ divides $N-1$. For, if not, let $N-1 = n\epsilon + \delta$. Then since

$$a^{N-1} = a^{n\epsilon+\delta} \equiv 1 \pmod{N},$$

it follows that $a^\delta \equiv 1 \pmod{N}$, which is impossible, since $\delta < \epsilon$. But ϵ does not divide m_i, for if it did, we would have $a^{m_i} \equiv 1 \pmod{N}$. Now let us write

$$\epsilon = n p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t},$$

where β_i may be zero and n is free of p_i. Consider the quotient

$$\frac{N-1}{\epsilon} = \frac{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}}{n p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}}.$$

Since $N-1$ is divisible by ϵ, we see that $n = 1$, and further,

$$\beta_1 \leqq \alpha_1, \ \beta_2 \leqq \alpha_2, \ \cdots, \ \beta_t \leqq \alpha_t.$$

Now the quotient

$$\frac{m_1}{\epsilon} = \frac{p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}}{p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}}$$

is not an integer, but it is identical with the preceding quotient except for the p_1's, so that we can write $\beta_1 > \alpha_1 - 1$, and in general $\beta_i > \alpha_i - 1$. But we have seen that $\beta_i \leqq \alpha_i$. Hence $\beta_i = \alpha_i$ and $\epsilon = N-1$. Finally let $\phi(N)$ be the totient of N. Then $a^{\phi(N)} \equiv 1 \pmod{N}$. Now if N were composite,

we would have $\phi(N)<N-1$. But this is impossible since $N-1=\epsilon$, and this is the smallest value of x satisfying $a^x\equiv 1 \pmod N$. Hence N is a prime.

Although this theorem is an improvement on Theorem 1, it still has the third defect of that theorem; that is, it tells us nothing of the character of a number N which satisfies the first part of the hypothesis but not the second. We have this situation, for example, whenever a is a quadratic residue of the prime N. It is clear that this defect cannot be eliminated by any theorem having the general form of Theorems 1 and 2. With this in mind we change our point of attack, and, by introducing a new element in the hypothesis, arrive at the following theorem.

THEOREM 3. *If* $a^x\equiv 1 \pmod N$ *for* $x=N-1$ *and if* $a^x\equiv r>1$ *for* $x=(N-1)/p$, *and if* $r-1$ *is prime to* N, *then all the prime factors of* N *are of the form* $np^\alpha+1$, *where* α *is the highest power to which the prime* p *occurs as a divisor of* $N-1$.

Let $N-1=qp^\alpha=mp$. Then we have for an hypothesis:

$$a^{N-1}\equiv 1 \pmod N, \tag{1}$$

$$a^m\equiv r \pmod N, \tag{2}$$

$$r-1 \text{ is prime to } N. \tag{3}$$

Now let κ be any prime factor of N and let ξ be the smallest value of x such that $a^x\equiv 1 \pmod \kappa$. Then since $a^{N-1}\equiv 1 \pmod \kappa$, we can show as in the preceding theorem that ξ divides $N-1$. Also by Fermat's theorem $a^{\kappa-1}\equiv 1 \pmod \kappa$. Hence ξ divides $\kappa-1$. But ξ does not divide m, for if it did we would have $a^m\equiv 1 \pmod \kappa$, which would imply that $r-1$ is divisible by κ. But $r-1$ is prime to N, and hence is prime to κ. Let us write $\xi=n_1p^\beta$, where n_1 is prime to p, and β may be zero. Consider the quotient

$$\frac{N-1}{\xi}=\frac{qp^\alpha}{n_1p^\beta}.$$

Since q is also prime to p, we have $\beta \leqq \alpha$. Now the quotient

$$\frac{m}{\xi} = \frac{qp^{\alpha-1}}{n_1 p^{\beta}}$$

is not an integer, since m is not divisible by ξ. Hence $\beta > \alpha - 1$. That is, $\beta = \alpha$ and $\xi = n_1 p^{\alpha}$. But since ξ divides $\kappa - 1$, we have $\kappa - 1 = n_2 \xi = n_1 n_2 p^{\alpha}$. Hence $\kappa = np^{\alpha} + 1$.

This theorem is the basis of a method given by Pocklington,* the importance of which seems to have escaped attention. In order to remove the final defect of Theorem 1, we must discuss the cases in which the various parts of the hypothesis are not fulfilled. If $a^{N-1} \equiv 1 \pmod{N}$, then N is composite, and its factors are not obtainable by this method. If $a^{m} \equiv 1 \pmod{N}$, all the factors of N divide the number $a^{m} - 1$, whose primitive factors are of the form $2nm + 1$. It is usually too troublesome, however, to try to show that N is a primitive factor. In practice, it is better to take m/p_2 or m/p_1^2 for a new value of m. If then $a^{m} \not\equiv 1$, all the factors of N are of the form $np_2^{\alpha_2} + 1$ or $np_1^{\alpha_1 - 1} + 1$, respectively. If $p_1 = 2$, this last case may be investigated without further calculation. However, this failure of the second part of the hypothesis is very rare, especially when care is taken in choosing the base a. Finally, if $r - 1$ is not prime to N, their G.C.D. will be a factor of N.

Suppose that we have applied Theorem 3 to a particular N and we find that $\kappa = np_1^{\alpha_1} + 1$. Then N is a prime if $p^{\alpha_1} > N^{1/2}$. If on the other hand $p_1^{\alpha_1}$ is too small, other factors of $N - 1$ may be dealt with in the same manner, so that we obtain $\kappa = np_1^{\alpha_1} p_2^{\alpha_2} \cdots + 1$, a form sufficiently exclusive to meet our needs. Suppose we have found in this way that $\kappa = nP + 1$, where P is the product of certain odd prime factors of $N - 1$. Our first inclination is to let n take on successive integral values, beginning with unity, and then try as divisors of N the prime values of κ. If $N^{1/2} > 10^{7}$ however, κ will run above the present limit of the list of primes, and we must

* Proceedings of the Cambridge Philosophical Society, vol. 18 (1914–16), p. 29.

try composite values of κ as well. Of course, about 5/6 of these are easily seen to be multiples of small primes; as for the others, it is easier actually to try them as divisors. All this may entail considerable labor, which may, for the most part, be obviated by seeking to express N as the difference of two squares. If every factor of $N=a^2-b^2$ is of the form $nP+1$, where P is odd, it follows that*

$$a^2 \equiv N \pmod{P^2}, \tag{1}$$

$$a \equiv 1 \pmod{P}. \tag{2}$$

One of the two values of a involved in (1) is eliminated by (2) so that a is restricted to one case in P^2. In fact, if

$$N \equiv 1 + kP \pmod{P^2},$$

then

$$a \equiv 1 + \frac{k}{2} P \pmod{P^2},$$

where, if k is odd, $k/2$ is taken modulo P. If the factors of N are, at the same time, of the form $n2^\lambda+1$, a similar restriction may be set up modulo $2^{2\lambda-1}$, which may be useful when λ is large. Suppose it is known that no factor of N is less than some limit W. Then a has the following range of values:

$$N^{1/2} < a < \frac{1}{2}\left(W + \frac{N}{W}\right).$$

In our case W is at least as large as $2P$, so that we have

$$N^{1/2} < a < \left(P + \frac{N}{4P}\right).$$

Although a has a greater range than κ, the possible values of a are more restricted especially when P is large. When P is small, so that there are a great number of values of either a or κ to be tried, use can be made of the "movable strip" method† of combining linear forms, which again favors the

* Lawerence, Quarterly Journal, vol. 28 (1896), pp. 285–311. Kraitchik, *Théorie des Nombres*, Paris, 1922, p. 146.

† Kraitchik, loc. cit., Chapter 2; also see Lawerence, loc. cit.

search for a. It is sometimes advantageous to use both methods. The direct method may be used to increase the limit W, thus decreasing the range for a, so that the work may be easily finished by the difference of squares.

Of the numbers $y^n \pm 1$, those for which $y=2$ have been most extensively investigated. Perhaps $y=10$ comes second in importance. If the results with 10^n-1 are comparatively few, it is not to be attributed to lack of interest, but rather to the lack of adequate methods with which to deal with these larger numbers. The preceding theory is most easily applied to divisors of $10^n \pm 1$, because of the ease with which the calculations may be performed.

Instead of employing the modulus N in our work, we may use any multiple of N; and, having found the residue of a^{N-1} modulo kN, we have only to divide the result by N to obtain the desired result. If N is a divisor of $10^n \pm 1$, we may choose k such that $10^n \pm 1 = kN$. The advantage in using the modulus $10^n \pm 1$ is that the division by this modulus may be performed by a single subtraction or addition. To cast 10^n+1 out of a number of $2n$ digits, we separate it into periods of n digits each, beginning at the right, and subtract the second period from the first. For 10^n-1, we simply add the two periods. An added advantage in using a composite modulus is that it provides an easy method of checking the work. We may select some small divisor of k, such as δ, and then construct an auxiliary table of $a^x \pmod{\delta}$, containing at most $\phi(\delta)$ entries. This table enables us to predict what any particular residue of $a^x \pmod{kN}$ should be congruent to $\pmod{\delta}$, by finding what x is congruent to $\pmod{\phi(\delta)}$. The calculation of the residue of a^{N-1} may be most easily performed as follows. We first make a table of exponents beginning with $n-1$, in which each entry is obtained by writing in the greatest integer in half the entry just above. This table has roughly $3 \log_{10} N$ entries, the final one being unity. Starting now at the bottom, we make a table of powers of a, each line being obtained by taking the square of the preceding entry, modulo $10^n \pm 1$, and multiplying this result

by a, whenever the desired entry corresponds to an odd exponent. On arriving at the top of the table, we cast N out of the final result and obtain $a^{N-1} \pmod N$. Of course the same procedure applies to the calculation of the residue of a^m.

Below are the results of applying Theorem 3 to seven divisors of $10^n \pm 1$ for $n = 19, 20, 23, 24, 27$, and 31. Besides the check already mentioned, every step in the following work has been verified by casting out multiples of 1001. It is confidently believed that, in those cases where $a^{N-1} \not\equiv 1$, this result is not due to a slip in the computation. However, the actual residues are given in hopes that they may prove useful to future workers who may wish to verify the results obtained.

1. *Example* 1. $N = 440,334,654,777,631$.

This number is the residuary factor of $10^{27}-1$. In fact

$$10^{27}-1 = 3^5 \cdot 37 \cdot 757 \cdot 333667 \cdot N.$$

The discovery of the factors 3 and 757 by congruence tables makes the factorization of $N-1$ a rather serious problem. At first sight we have

$$N-1 = 2 \cdot 3^3 \cdot 5 \cdot 1,630,869,091,769.$$

The first two factors are of no use, since we know in advance that every factor of N is of the form $54n+1$. After examining the large factor for divisors less than 1000 without result, 5 was chosen as p^a; it was found that

$$3^{(N-1)/5} = 3^m \equiv 31343325933897 = r \pmod N,$$

and further that

$$r^5 \equiv 3^{N-1} \equiv 1 \pmod N,$$

$r-1$ being prime to N. Every factor of N is then of the form $5n+1$. It was next decided to undertake the arduous though not impossible task of seeking to represent N as the difference of two squares, taking for W the limit 120,000 set by congruence tables. Before actually starting the work, however, a further attempt to factor the number 1,630,869,091,769

was made by means of the "factor stencils" of D. N. Lehmer,* which, though still incomplete, are fortunately available to the present author. Although this number of unknown form is nearly 1000 times larger than the intended limit of the stencils, it was found without great difficulty that

$$1,630,869,091,769 = 31249 \cdot 52,189,481.$$

Choosing 52189481 as a new value of p^α, it was easily found that

$$3^m \equiv 78533825886276 = r \pmod{N},$$

$r-1$ being prime to N. The factors of N are then of the form $52189481n+1$. But $N^{1/2}$ is less than 20984153. Hence N is a prime and we have the complete factorization

$$10^{27} - 1 = 3^5 \cdot 37 \cdot 757 \cdot 333667 \cdot 440334654777631.$$

2. *Example* 2. $N = 9,999,000,099,990,001.$

This number is $(10^{20}+1)/(10^4+1.)$ It was found that

$$3^{N-1} \equiv 3703264653988674 \pmod{N}.$$

Hence N is composite. This number has been examined for factors less than its cube root without success. It is therefore the product of two primes.

3. *Example* 3. $N = 9,999,999,900,000,001.$

This number is $(10^{24}+1)/(10^8+1)$. In this case we have

$$N - 1 = \frac{10^{24}+1}{10^8+1} - 1 = \frac{10^8}{10^8+1}(10^{16} - 1)$$
$$= 2^8 \cdot 5^8 \cdot 3^2 \cdot 11 \cdot 73 \cdot 101 \cdot 137.$$

The divisors 2^8 and 5^8 were chosen as values of p^α and tested in one operation as follows. It was first found that

$$7^{(N-1)/10} \equiv 7128121476353673 = r \pmod{N}.$$

Then

$$r^2 \equiv 7^{(N-1)/5} \equiv 428233546143224 \pmod{N},$$

* Proceedings of the National Academy, vol. 11 (1925), p. 97.

and finally

$$r^5 \equiv -1 \quad \text{and} \quad r^{10} \equiv 1 \pmod{N}.$$

Since r^2-1 is prime to N, it follows that every factor of N is of the form 5^8n+1 and also of the form 2^8n+1. But $N^{1/2}<10^8$, so that N is a prime and we have the complete factorization:

$$10^{24}+1 = 17 \cdot 5882353 \cdot 9999999900000001.$$

4. *Example* 4. $N = 909,090,909,090,909,091.$

This number is $(10^{19}+1)/11$. It was found that

$$3^{N-1} \equiv 1 \pmod{N}.$$

Now we have

$$N-1 = \frac{10^{19}+1}{10+1} - 1 = \frac{10}{11}(10^{18}-1)$$
$$= 2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \cdot 52579 \cdot 333667.$$

Choosing as p the number 333667, we obtain

$$3^m \equiv 314776999832050172 = r \pmod{N},$$

$r-1$ being prime to N. Every factor of N is then of the form $333667n+1$, and also $19n+1$; or, in other words, $6339673n+1$. Using this restriction, we might test directly for the factors of N with n ranging from 1 to 150. About 125 of these values of κ might be ruled out as being multiples of small primes. The remaining 25 would have to be tested. Instead, let us write $N=a^2-b^2$. Then since

$$N \equiv 1 + 56743 \cdot 6339673 \pmod{(6339673)^2},$$

it follows that

$$a \equiv 1+3202708 \cdot 6339673 \pmod{(6339673)^2},$$

or

$$a = 4019453746929n + 20304121434485. \tag{1}$$

Taking $2P$ or 12679346 as a value of W, we find that a has the following range:

$$933462589 < a < 35855622005. \tag{2}$$

The smallest value of a allowed by (1) is more than 500 times larger than the largest value allowed by (2). Hence a does not exist and N is a prime. We then have the complete factorization

$$10^{19} + 1 = 11 \cdot 909090909090909091.$$

5. *Example* 5. $N = 999,999,999,000,000,001.$

This number is $(10^{27}-1)/(10^9-1)$. In this case it was found that

$$3^{N-1} \equiv 437299180785434949 \pmod{N};$$

N is therefore composite. This number and the third one examined above fill in two entries in the following rather curious table:

91	Composite
9901	Prime
999001	Composite
99990001	Prime
9999900001	Composite
999999000001	Prime
99999990000001	Composite
9999999900000001	Prime
999999999000000001	Composite

Unfortunately, the next entry in this table is composite.

6. *Example* 6. $N = 11,111,111,111,111,111,111,111.$

This number is $(10^{23}-1)/9$. It was found that

$$3^{N-1} \equiv 1268486354649455149380 \pmod{N}.$$

Hence we know that N is composite. This number is listed as a prime in a table of divisors of $10^n \pm 1$ given by Loof.* In a French edition† of this table is given the curious entry: 11111 · 11111? Bickmore‡ marks this number as a prime

* Archiv der Mathematik und Physik, vol. 16 (1851), pp. 54–57.

† Nouvelles Annales, vol. 14 (1855), pp. 115–117. Compare Messenger of Mathematics, vol. 33 (1903–04), p. 96.

‡ Nouvelles Annales, (3), vol. 15 (1896), p. 222.

on the authority of Loof. The character of $(10^{23}-1)/9$ as revealed for the first time by the above result will be settled once and for all only when the factors are discovered. The numbers made up entirely of the digit 1 are of interest on account of the rarity of primes among them. Only those numbers that have a prime number of digits have any chance to be primes, and this set may be said to correspond to the Mersenne numbers. Besides the obvious cases 1 and 11, this former set contains but one other prime less than the limit $(10^{37}-1)/9$. This prime is $(10^{19}-1)/9$ and was examined independently by Hoppe and Kraitchik by the difference of two squares. By way of comparison with the Mersenne numbers, it is found that there are no less than 9 primes less than $2^{37}-1$. The numbers $10^{2^n}+1$, which correspond to the Fermat numbers, also show a scarcity of primes. The only primes that have been found in this set are 11 and 101, up to the limit $10^{64}+1$.

7. *Example* 7. $N = 909,090,909,090,909,090,909,090,909,091$.

This number is $(10^{31}+1)/11$. In this case it was found that

$$3^{N-1} \equiv 1 \pmod{N}.$$

Now we have

$$N-1 = \frac{10^{31}+1}{10+1} - 1 = \frac{10}{11}(10^{30}-1)$$
$$= 2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 37 \cdot 41 \cdot 211 \cdot 241 \cdot 271 \cdot 2161 \cdot 9091 \cdot 2906161.$$

Here $N-1$ breaks up into so many small factors that no single one used for p gives sufficient restriction. Selecting $p_1 = 2906161$ and $p_2 = 9091$, it was found that

$$3^{m_1} \equiv 40421597451230127586416772 1473 = r_1 \pmod{N},$$

and

$$3^{m_2} \equiv 55433011245022437225001034 8117 = r_2 \pmod{N},$$

r_1-1 and r_2-1 being prime to N. Hence the factors of N belong to the forms

$$\left.\begin{matrix} 31n+1 \\ 9091n+1 \\ 2906161n+1 \end{matrix}\right\} \text{ or } 819017199181n+1.$$

Now $N^{1/2}=953462589245592$. To test directly for the factors of N would require 1164 values of n. Instead, we find that

$$N \equiv 1 + 645945952735 \cdot 819017199181 \pmod{(819017199181)^2},$$

so that

$$a \equiv 1 + 732481575958 \cdot 819017199181 \pmod{(819017199181)^2}$$

or

$$(1) \qquad a = 670789172554289827070761n + 599915008792806066890399.$$

To find the range for a, we let $W=2P=1638034398362$ and obtain

$$(2) \qquad 953462589245592 < a < 554988900110999445.$$

We see that the smallest value of a in (1) is more than a million times larger than the largest value in (2). Hence a does not exist and N is a prime. We then have the complete factorization

$$10^{31}+1 = 11 \cdot 909090909090909090909090909091.$$

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THE UNIVERSITY OF CALIFORNIA

ON ANALYTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN THE NEIGHBORHOOD OF NON-ANALYTIC SINGULAR POINTS*

BY B. O. KOOPMAN†

We shall consider the system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} \tag{1}$$

in the complex neighborhood of a point, say $(0, 0, \cdots, 0)$, at which the X's vanish simultaneously. We shall suppose throughout that

$$X_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + \phi_i(x_1, \cdots, x_n), \qquad (i = 1, 2, \cdots, n),$$

and that the general solutions of the related linear system

$$\frac{dx_1}{a_{11}x_1 + \cdots + a_{1n}x_n} = \cdots = \frac{dx_n}{a_{n1}x_1 + \cdots + a_{nn}x_n} \tag{2}$$

approach zero, without being identically zero.‡ Consider a solution of the equations (2). When will there exist an analytic solution of the equations (1) which behaves essentially like this near $(0, 0, \cdots, 0)$? How many such solutions will there be? Our object is to answer these questions without restricting the functions ϕ_i, except as regards their behavior in a certain region of the complex $x_1x_2 \cdots x_n$-space abutting on the origin, in which the given solution of (2) is embedded. We shall thus obtain a treatment less restricted than those of Briot and Bouquet, Poincaré, Picard, Dulac, Horn, etc., who assume the ϕ's to be convergent power series of terms of degree higher than the first.§

* Presented to the Society, October 30, 1926.

† National Research Fellow in Mathematics.

‡ See, however, the concluding note of this paper.

§ For full references, see J. Malmquist, *Sur les points singuliers des équations différentielles*, Arkiv för Matematik, Astronomi och Fysik, vol. 15, No. 3, (1920), pp. 1–5.

It is necessary first of all to reduce the system to a standard form. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the roots of the equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0,$$

and $r_1, r_2 \cdots, r_\nu$ their respective multiplicities ($r_1+\cdots+r_\nu=n$). Then we may, according to the classical theory,* transform the equations (1) by means of a linear substitution of the x's so that the expressions for the new X's in terms of the new x's become

$$(3)\quad \begin{cases} \lambda_1 x_1+\phi_1,\ \lambda_1 x_2+x_1+\phi_2,\ \cdots,\lambda_1 x_{r_1}+x_{r_1-1}+\phi_{r_1}, \\ \lambda_2 x_{r_1+1}+\phi_{r_1+1},\ \lambda_2 x_{r_1+2}+x_{r_1+1}+\phi_{r_1+2},\ \cdots, \\ \qquad\qquad \lambda_2 x_{r_1+r_2}+x_{r_1+r_2-1}+\phi_{r_1+r_2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{cases}$$

It is understood here, as in all cases where a transformation of the x's is made, that the new ϕ's denote that part of the new X's which contains the old ϕ's. Now the general solutions of the new equations (2) corresponding to the new values (3) of the X's are of the form†

$$(4)\quad \begin{cases} x_i=\left(C_1\dfrac{t^{i-1}}{(i-1)!}+C_2\dfrac{t^{i-2}}{(i-2)!}+\cdots\right. \\ \qquad\left.+C_{i-1}t+C_i\right)e^{\lambda_1 t},\quad (i=1,2,\cdots,r_1), \\ x_{r_1+i}=\left(C_1'\dfrac{t^{i-1}}{(i-1)!}+C_2'\dfrac{t^{i-2}}{(i-2)!}+\cdots\right. \\ \qquad\left.+C_{i-1}'t+C_i'\right)e^{\lambda_2 t},\quad (i=1,2,\cdots,r_2), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{cases}$$

* See e. g., E. Goursat, *Cours d'Analyse Mathématique*, Paris, Gauthier-Villars, 1918, vol. II, p. 495.

† See Goursat, loc. cit.

Here, dt is the common ratio in (2). With the aid of these expressions it is easy to prove that, in the case we are considering, where the general solution of (2) approaches $(0, 0, \cdots, 0)$, the points λ_i all lie on the side away from the origin of a straight line, l: $\arg \lambda = c$, drawn in the λ-plane, not through the origin. Now the transformation of the x's which leads to the expressions (3) may be so conducted that $\lambda_1' = 1$, while the real parts of $\lambda_2, \cdots, \lambda_\nu$ are not less than 1. This is accomplished by dividing the X's through by the appropriate $\lambda' e^{c-\pi/2}$, say $\lambda_1' e^{c-\pi/2}$, at a certain stage of the transformation. Furthermore, the effect of performing the substitution

$$x_{r_1+1} = x_1 x_{r_1}' + 1, \quad x_{r_1+2} = x_1 x_{r_1}' + 2, \cdots, \quad x_{r_1+r_2} = x_1 x_{r_1}' + r_2,$$

is to replace the values in (3) by similar expressions in which λ_2 is replaced by $\lambda_2 - \lambda_1$. Thus we may assume that the real parts of $\lambda_2, \cdots, \lambda_\nu$ are all less than 2.

Consider then the system

$$\frac{dx_i}{dt} = X_i, \qquad (i = 1, 2, \cdots, n) \tag{1'}$$

where the X's have the values (3), and where

$$1 \leqq \lambda_i' < 2, \quad (\lambda_i = \lambda_i' + (-1)^{1/2}\lambda_i''), \quad (i = 1, 2, 3, \cdots, \nu).$$

Let there be given a special solution:

$$x_i = f_i(t), \qquad (i = 1, 2, \cdots, n) \tag{4'}$$

of the corresponding linear equations (2); i. e., let known values of the constants C in equations (4) be given. Let T be the semi-infinite band in the t-plane defined by the inequalities $T: -\infty < t' \leqq 0$, $-h < t'' < h$, where $t = t' + (-1)^{1/2} t''$, and h is some positive constant. By means of the equations (4') the strip T goes over into a two-dimensional figure in the complex $2n$-dimensional $x_1 x_2 \cdots x_n$-space; as the point t moves out to infinity in T, the corresponding point $(x_1, x_2, \cdots, x_n)$ approaches $(0, 0, \cdots, 0)$. Now consider the $2n$-dimensional region R of the complex $x_1 x_2 \cdots x_n$-

space corresponding to the given positive constants ρ, $\delta(<2-\lambda_i')$:

$$R: |x_i| < \rho\,; \quad |x_i - f_i(t)| < e^{(2-\delta)t'}\,; \quad (i = 1,2, \cdots, n).$$

Here t is supposed to remain in T. Our hypotheses with regard to the functions ϕ_i are as follows:

(*a*) *They are single-valued and analytic in the region R.*

(*b*) *All their first partial derivatives approach zero as* $(x_1, x_2, \cdots, x_n)$ *approaches* $(0, 0, \cdots, 0)$ *through points of R.*

(*c*) *The* ϕ*'s approach zero as the square of the distance of the point* $(x_1, x_2, \cdots, x_n)$ *from* $(0, 0, \cdots, 0)$, *regarded as points in 2n-dimensional space. Analytically, this means that there exists a constant K such that*

$$|\phi_i(x_1, \cdots, x_n)| < K(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2), \quad (i = 1,2, \cdots, n), \tag{5}$$

for all $(x_1, x_2, \cdots, x_n)$ *in R.*

Under these hypotheses we shall prove that *the equations* (1′) *admit the solutions* $x_i = F_i(t)$, $(i=1, 2, \cdots, n)$, *single-valued and analytic in a certain sub-region T′ of T,* $T' : -\infty < t' < \Gamma \leqq 0$, $-h < t'' < h$, *and approaching zero as* $t' = -\infty$. *Furthermore, that there is a positive constant B and an arbitrarily small constant* γ *such that* $|F_i(t) - f_i(t)| < Be^{(2-\gamma)t'}$, $(i=1, 2, \cdots, n)$; *and that* $F_i(t)$ *is the only solution having these properties.*

If the ϕ's were known functions of the time, the knowledge of the solutions (4) would lead to the explicit solution of (1′) in terms of definite integrals by means of the classical methods. In the actual case, the formal process will yield integral equations which are equivalent to the equations (1′). These will be solved by the method of successive approximations.*

* The idea of extending a solution of equations like (2) to the equations (1) by the method of successive approximations has been made use of by I. Bendixon (Acta Mathematica, vol. 24 (1900)) and, in a form quite

The hypothesis (b) implies that the functions ϕ_i satisfy a Lipschitz condition in the region R:

$$(6)\qquad \begin{aligned} &|\phi_i(x_1^1, \cdots, x_n^1) - \phi_i(x_1^2, \cdots, x_n^2)| \\ &\qquad < A(|x_1^1 - x_1^2| + \cdots + |x_n^1 - x_n^2|), \end{aligned}$$

and that *the constant A can be taken as small as we like provided we take ρ sufficiently small.* In the usual proof of (6) a "convex" region is assumed, in which the points (x^1) and (x^2) can always be joined by a straight line segment lying wholly in its interior.* It will be possible to apply the method to our case if we can show that the given points may always be joined by a regular curve lying in R, the ratio of the length of which to the distance between the points being less than a fixed constant independent of (x^1) and (x^2). A little computation, with the aid of the expressions (4), will show that, if ρ be sufficiently small, the curve defined below has the requisite properties. Suppose that

$$\begin{aligned} &|x_1^1|^2 + |x_2^1|^2 + \cdots + |x_n^1|^2 \\ &\qquad \leqq |x_1^2|^2 + |x_2^2|^2 + \cdots + |x_n^2|^2; \end{aligned}$$

through the point (x^1) pass the curve obtained by setting $t'' = 0$ in equations (4) (this involves making a proper choice of the C's). Let $(\bar{x}^2)$ be its point of intersection with the hypersphere through (x^2):

$$\begin{aligned} &|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \\ &\qquad = |x_1^2|^2 + |x_2^2|^2 + \cdots + |x_n^2|^2. \end{aligned}$$

Now the curve we want is composed of the straight line joining (x^2) with $(\bar{x}^2)$, and of the arc of the preceding curve intercepted by (x^1) and $(\bar{x}^2)$. The remainder of the proof

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similar to our own, by O. Perron (Mathematische Zeitschrift, vol. 14, (1922)). These authors were studying the geometric nature of the integral curves at a singular point in the real xy-plane, and were not concerned with the complex domain.

* See E. Goursat, loc. cit., p. 380.

consists in examining a line integral taken along this curve, and applying the hypothesis (b).*

We turn now to the proof of the existence theorem. We shall give the details of the proof for a system of the second order, leaving the obvious extension to the reader. The first typical case is that of the system†

$$(1'') \qquad \frac{dx}{dt} = x + \phi(x,y), \qquad \frac{dy}{dt} = \lambda y + \psi(x,y),$$

where $1 \leqq \lambda' < 2$, and where ϕ and ψ verify the hypotheses (a), (b), (c) with respect to the solution

$$(4'') \qquad x = ae^t, \qquad y = be^{\lambda t}.$$

The corresponding integral equations are found to be

$$(7) \qquad \begin{aligned} x &= ae^t + \int_{-\infty}^{t} \phi(x,y)e^{t-\xi}d\xi, \\ y &= be^{\lambda t} + \int_{-\infty}^{t} \psi(x,y)e^{\lambda(t-\xi)}d\xi, \end{aligned}$$

where ξ is a variable of integration: $x = x(\xi)$, $y = y(\xi)$ in the integrand. We have chosen $-\infty$ as the lower limit, since we are interested in solutions which differ from $(4'')$ by an amount which vanishes in comparison with e^t as $t' = -\infty$. The path of integration could be any curve running from the given value t out to infinity in the strip T; but it is desirable to fix it definitely, and we shall take it to be the horizontal ray running from t to the left indefinitely, i. e., $-\infty < \xi' \leqq t$, $\xi'' = t''$.

Let the successive approximations be defined by recurrence as follows:

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* See Goursat, loc. cit., p. 380. We use the finiteness of the logarithmic spiral, and such curves.

† For simplicity in writing, we are taking the imaginary part of the coefficient of x in $(1'')$ zero; but the proof in the general case is quite similar.

$$x_0 = ae^t, \qquad x_{k+1} = ae^t + \int_{-\infty}^{t} \phi(x_k, y_k) e^{t-\xi} d\xi \,;$$

$$y_0 = be^{\lambda t}, \qquad y_{k+1} = be^{\lambda t} + \int_{-\infty}^{t} \psi(x_k, y_k) e^{\lambda(t-\xi)} d\xi \,;$$

$$(k = 0,1,2,3, \cdots).$$

We have to show that the functions x_k and y_k all exist and are analytic, and approach limits which are the required solutions as $k = \infty$. Writing as usual

$$u_k = x_k - x_{k-1}, \qquad v_k = y_k - y_{k-1}, \qquad (k = 1,2,3, \cdots),$$

we have

$$x_k = x_0 + u_1 + u_2 + \cdots + u_k,$$

$$y_k = y_0 + v_1 + v_2 + \cdots + v_k,$$

and we must examine the series

$$\begin{cases} u_1 + u_2 + u_3 + \cdots \\ v_1 + v_2 + v_3 + \cdots \end{cases} \tag{8}$$

for convergence.

From the expression

$$u_1 = \int_{-\infty}^{t} \phi(x_0, y_0) e^{t-\xi} d\xi \,,$$

and the hypothesis (c), we have

$$|u_1| \leqq \int_{-\infty}^{t'} |\phi(x_0, y_0)| e^{t'-\xi'} d\xi', \qquad (\xi = \xi' + (-1)^{1/2}\xi''),$$

$$< K \int_{-\infty}^{t'} (|x_0|^2 + |y_0|^2) e^{t'-\xi'} d\xi'$$

$$< Ke^{t'} \int_{-\infty}^{t'} (|a|^2 e^{\xi'} + |b|^2 e^{(2\lambda'-1)\xi' - 2\lambda''t''}) d\xi'$$

$$< K\left(|a|^2 + \frac{|b|^2}{2\lambda' - 1} e^{2(\lambda'-1)t' - 2\lambda''t''}\right) e^{2t'}.$$

Similarly,

$$(9) \qquad |v_1| < K\left(\frac{|a|^2}{2-\lambda'} + \frac{|b|^2}{\lambda'} e^{2(\lambda'-1)t'-2\lambda''t''}\right) e^{2t'}.$$

Hence it is possible to find a positive constant M such that

$$|u_1| < Me^{2t'}, \qquad |v_1| < Me^{2t'}.$$

Moreover, the preceding computation establishes the existence and analyticity of the functions u_1, v_1 in T, as it follows from the elementary considerations of the uniform convergence of definite integrals.

For the further u's and v's we have the equations

$$u_{k+1} = \int_{-\infty}^{t} [\phi(x_k, y_k) - \phi(x_{k-1}, y_{k-1})] e^{t-\xi} d\xi ,$$

$$v_{k+1} = \int_{-\infty}^{t} [\psi(x_k, y_k) - \psi(x_{k-1}, y_{k-1})] e^{\lambda(t-\xi)} d\xi ,$$

$$(k = 1, 2, 3, \cdots).$$

Suppose that all the functions up to and including x_k, y_k exist and are analytic in the region T, that they map T in the interior of R, and, finally, that there exist a positive constant H_k for which $|u_k| < H_k e^{2t'}$, $|v_k| < H_k e^{2t'}$. It will then follow from hypothesis (b) that

$$|u_{k+1}| \leqq \int_{-\infty}^{t'} |\phi(x_k, y_k) - \phi(x_{k-1}, y_{k-1})| e^{t'-\xi'} d\xi'$$

$$< A \int_{-\infty}^{t'} (|u_k| + |v_k|) e^{t'-\xi'} d\xi'$$

$$< 2AH_k e^{t'} \int_{-\infty}^{t'} e^{\xi'} d\xi' = 2AH_k e^{2t'}. \qquad [\text{See } (6).]$$

Similarly,

$$(10) \qquad |v_{k+1}| < \frac{2AH_k}{2-\lambda'} e^{2t'}.$$

Thus, if G denote the greater of the two quantities 2 and $2/(2-\lambda')$, we have

$$|u_{k+1}| < AG \cdot H_k e^{2t'}, \qquad |v_{k+1}| < AG \cdot H_k e^{2t'}.$$

Moreover, u_{k+1} and v_{k+1} exist and are analytic in T, and likewise for x_{k+1} and y_{k+1}.

We wish to know when the locus of the corresponding point, (x_{k+1}, y_{k+1}), will lie inside R. We have

$$\begin{aligned} |x_{k+1} - ae^t| &\leqq |u_1| + |u_2| + \cdots + |u_{k+1}| \\ &< (1 + AG + A^2G^2 + \cdots + A^kG^k)Me^{2t'}, \end{aligned}$$

the last inequality resulting from the repeated application of the results of the two preceding paragraphs. Now consider the geometric series $1+AG+A^2G^2+A^3G^3+\cdots$. *We shall suppose that* ρ *is taken so small that* $AG<1$; this is permissible in view of what has been deduced from hypothesis (b); and in order to be sure that the loci of $(x_0, y_0), \cdots, (x_k, y_k)$ lie in this region, it is only necessary to confine t to a suitable region T': $t'<\Gamma$. Then the geometric series will converge, and have the sum $1/(1-AG)$. Hence

$$(11) \quad |x_{k+1} - ae^t| < \frac{M}{1-AG} e^{2t'}, \quad |y_{k+1} - be^{\lambda t}| < \frac{M}{1-AG} e^{2t'}.$$

A comparison of these inequalities with those defining R shows that if Γ in the definition of T' be given a sufficiently small value (which will be independent of k), and if t be confined to T', the locus of (x_{k+1}, y_{k+1}) will lie wholly within R.

Thus, our points (x_i, y_i) will never leave R as long as t is in T', our terms will all be determined by recurrence, and the above inequalities will apply to them. They will be single-valued and analytic for t in T', and the series will be dominated by the convergent geometric series. It follows by the usual reasoning that (x_i, y_i) converges to (x, y), the required solutions of (7) and hence of (1″).

The inequalities (11) evidently yield the result

$$|x - ae^t| < Be^{2t'}, \qquad |y - be^{\lambda t}| < Be^{2t'}.$$

Now there can not be more than one solution which is thus related to $(ae^t, be^{\lambda t})$. For if $\bar{x}$, $\bar{y}$ were a second solution, it would satisfy (7), and it would follow by previous formulas that

$$\bar{x} - x_{k+1} = \int_{-\infty}^{t} [\phi(\bar{x}, \bar{y}) - \phi(x_k, y_k)]e^{t-\xi}d\xi, \text{ etc.}$$

Hence

$$|\bar{x} - x_{k+1}| < A\int_{-\infty}^{t'} (|\bar{x} - x_k| + |\bar{y} - y_k|)e^{t'-\xi'}d\xi', \text{ etc.}$$

It follows by recurrence, as before, that

$$|\bar{x} - x_k| < A^k G^k \cdot Be^{2t'}, \text{ etc.};$$

hence we have*

$$x \equiv \bar{x}, \qquad y \equiv \bar{y}, \qquad \text{(q.e.d.)}.$$

The form of the uniqueness theorem originally stated is proved in the same manner.

We turn now to the second typical case for the system of the second order. It is that of the equations

$$\frac{dx}{dt} = x + \phi(x, y), \qquad \frac{dy}{dt} = x + y + \psi(x,y).$$

The solution of the corresponding linear system is

$$x = ae^t, \qquad y = (at + b)e^t,$$

and the equivalent integral equations are

$$\begin{cases} x = ae^t + \displaystyle\int_{-\infty}^{t} \phi(x, y)e^{t-\xi}d\xi, \\ y = (a + bt)e^t + \displaystyle\int_{-\infty}^{t} [\psi(x, y) + (t - \xi)\phi(x, y)]e^{t-\xi}d\xi. \end{cases}$$

The solution by successive approximations is just the same as before, and the computation is performed on parallel lines. The inequality (9) is replaced by the following:

* See Goursat, loc. cit., p. 377.

$$|v_1| \leqq e^{t'} \int_{-\infty}^{t'} (|\psi_0| + |t-\xi| \cdot |\phi_0|) e^{-\xi'} d\xi'$$

$$< Ke^{t'} \int_{-\infty}^{t'} (|x_0|^2 + |y_0|^2)(1 + |t-\xi|) e^{-\xi'} d\xi'$$

$$< Ke^{t'} \int_{-\infty}^{t'} (|a|^2 + |a\xi + b|^2)(1 + |t-\xi|) e^{\xi'} d\xi'.$$

Hence, if we take t in a sufficiently restricted region T', we have

$$|v_1| < M'e^{t'} \int_{-\infty}^{t'} e^{\xi'-\gamma} d\xi' = Me^{(2-\gamma)t},$$

where γ is a positive constant which we may take as small as we wish, provided Γ be taken sufficiently small. Likewise, to prove the inequality corresponding with (10) we have

$$|u_k| < H_k e^{(2-\gamma)t'}, \qquad |v_k| < H_k e^{(2-\gamma)t'},$$

whence

$$|v_{k+1}| < e^{t'} \int_{-\infty}^{t'} A(|u_k| + |v_k|)(1 + |t-\xi|) e^{-\xi'} d\xi'$$

$$< AH_k e^{t'} \int_{-\infty}^{t'} (1 + |t-\xi|) e^{(1-\gamma)\xi'} d\xi'.$$

Here again, if t is taken in a region T', with Γ sufficiently small, we shall have

$$|v_{k+1}| < H_k \cdot AGe^{(2-\gamma)t'}.$$

We leave it to the reader to complete the proof.

Note: Although we have supposed throughout that the general solution of the equations (2) approaches zero as $t = \infty$, it will be seen that even in the contrary case, if a particular solution of (2) approaches zero, it is possible to frame a theorem, like the one which has been given, which establishes the extension of the particular solution to the given differential equations (1).

PRINCETON UNIVERSITY

CORRADO SEGRE

BY J. L. COOLIDGE

There is a pronounced rise and fall in the tide of mathematical interest in different countries and at different times. This is well shown in the case of geometry. In Great Britain and Ireland, in the Victorian era, Hamilton and Clifford, Salmon and Casey, Cayley and Sylvester, constituted a veritable English-Irish geometric school. With the deaths of these men, and the decline in the fashion for algebraic invariants, all of this efflorescence seemed to wither away, till the University of Oxford appointed to the Savillian professorship of geometry a man who had attained high distinction in the analytical theory of numbers. In France, the name of Chasles exercised, at one time, a fascination which we perhaps do not quite understand today. Darboux, with his masterly *Théorie Générale des Surfaces*, raised to himself a monument more enduring than bronze, but no geometric school arose therefrom, and the center of gravity of mathematical interest in France today is certainly not in the field of geometry. In Germany, the fluctuation has been distinctly less. In Italy, the interest in geometry has produced notable results. The classical differential geometry owes an immense debt to Beltrami and Bianchi; the Riemannian geometers take as their starting point the absolute calculus of Ricci and Levi-Civita; as for the modern birational geometry, that is almost a monopoly of Italian mathematicians and a few others like Macaulay and Snyder, who have been strongly influenced by Italian thought.

What is the reason for this state of affairs? It would be equally wrong to ascribe it entirely to chance or to seek a recondite explanation in the psychology of peoples. The real ground is to be found in the presence or absence of great teachers of geometry. Salmon was a great teacher indeed, but Dublin was not a good center for distributing the pure milk of the word. Darboux was a marvellous lecturer, but he lacked the personal qualities necessary to inspire youth. It is of Jules Tannery, not of Darboux, that the French mathematicians of today say: "C'était notre père à nous tous." In Germany, on the other hand, Klein's extraordinary inspiration was felt quite as much in geometry as elsewhere; Reye created an enthusiasm for pure projective geometry that went almost to excess; the present geometric activity in Hamburg and Bonn is directly traceable to the influence of Study. The geometric "risorgimento" in Italy is the result of the efforts of certain great teachers such as Cremona, Battaglini, D'Ovidio, and the subject of the present sketch.

Corrado Segre was born in Saluzzo, August 20, 1863*, and received his laureate at Turin when less than twenty years of age. He never left

* In preparing the present account, I have consulted, in addition to Segre's own writings, Castelnuovo, Rendiconti dei Lincei, (5a), vol. 33[2]

the Piedmontese capital thereafter, except for short periods; his active life was passed there as assistant and as professor. At some period during the early part of his career, he had a severe attack of brain fever brought on by overwork. He complained sadly that his capacity for labor was never the same thereafter. When one reflects that the number of his published titles is 128, it fatigues the imagination to speculate as to what he might have accomplished had he retained his early strength. He died unexpectedly, after an operation, on May 18, 1924.

The connecting thread which runs through Segre's work is the projective geometry of n dimensions. In the hands of a man of small grasp, this might easily degenerate into barren formalism. Not so in his case. He never lost sight of the relations between a general formulation and concrete problems, and never failed to familiarize himself with other methods of treating the question in hand, besides those which he personally preferred. Let us trace the line of his thought. The simplest figures in any space are the linear sub-spaces. Suppose first that we have a finite number of such. The obvious problems are those of incidence. The beginner has little idea of the complications which quickly arise in such questions. Is it at all obvious that in four-space exactly five planes will intersect six given lines in general position? Segre treats them in two papers separated by a considerable interval of time.* Next to linear spaces we have quadratic varieties. Segre's thesis is devoted to a study of the hyperquadric in n-space.† A single hyperquadric is of slight mathematical interest compared with the study of two such figures and the classification of their various types of intersection. The beautiful plane curve which Darboux has called a cyclic and the English and Irish have described as a bicircular quartic, is the stereographic projection on the projective plane of the intersection of the Gauss sphere with a general quadric surface. Ascending one dimension, the general cyclide or quartic surface with a double conic may be reached in a similar way by a projection in four-space. Segre devoted one of his earliest and largest papers to the classification of such surfaces.‡ Going up two more dimensions, the line-geometry of Plücker and Klein consists in identifying the lines of a projective three-space, with the points of a hyperquadric in five-space. The intersections of two hyperquadrics will correspond to the lines of a quadratic complex. Segre studies the general quadratic complex and some particular varieties, with special reference to the singular elements, most fruitfully by this method.§

(1924); Baker, Journal of the London Society, vol. 1 (1926), and especially Loria, Annali di Matematica, (4), vol. II (1925). Here will be found a complete bibliography of Segre's work.

* Rendiconti di Palermo, vol. 2 (1888); Rendiconti dei Lincei, (5), vol. 9^2 (1900).

† Torino Memorie, (2), vol. 36 (1883).

‡ Mathematische Annalen, vol. 34 (1884).

§ There are a number of papers on this topic. See Loria, loc. cit., pp. 14–15.

The problem of classifying the intersections of two hyperquadrics is best solved by the use of the elementary divisors of Weierstrass. An almost identical algebraic analysis will lead to the classification of linear transformations and their fixed elements. This problem also received Segre's attention.*

Reverting to linear spaces, Segre had the curious but fruitful idea of studying those systems of points which are in one to one correspondence with groups of points, one in each of a number of given linear spaces. The simplest example of this is the perfect correspondence between the individual points of a general quadric and pairs of points of two given lines. The case which particularly interested Segre was that where there were two conjugate imaginary linear varieties.† These considerations led him to another field to which, for a few years, he gave considerable attention. If a complex space of any number of dimensions be given, there are two questions which immediately suggest themselves. How can the points of this space, and of various loci therein, be represented by real geometrical figures? What are the geometrical characteristics of the loci of points whose coordinates are functions of a certain number of *real* parameters? For instance, how shall we best represent all the complex points of a plane? Given a system of points in a plane whose coordinates depend analytically on two real parameters, how shall we determine whether they generate a curve? Segre laid the foundation for the future study of the second of these problems and added considerably to our knowledge of the first.‡ He thereby established contact with Study, whom he subsequently met at the Heidelberg Congress in 1904, and with whom he retained cordial relations for many years. It must be confessed, however, that few mathematicians have been found who cared to enter this which Segre calls "Un nuovo campo di ricerche geometriche." Even so competent a critic as Baker§ fails completely to grasp the real significance of this part of Segre's work. Perhaps he felt some regret that so little was done to extend geometrical science in these directions. The present writer feels that regret keenly.

Let us return to linear spaces and suppose that we have not a finite but an infinite number of them. We are led, as the case may be, to problems of algebraic or of differential geometry. Suppose that we have a one-parameter system of lines in our space. If it be an algebraic system, it will correspond to a curve in 5-space, and so have a definite genus. But these lines will generate an algebraic surface with various genera, arithmetic, geometric, etc. On the other hand if the system be not algebraic, but still analytic, the lines will generate a surface. Is this surface de-

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* Rendiconti dei Lincei, (3), vol. 19 (1884); Torino Memorie, (2), vol. 37 (1885); Journal für Mathematik, vol. 100 (1886).

† Rendiconti di Palermo, vol. 5 (1891).

‡ Torino Atti, vol. 25 (1889), vol. 26 (1890); Mathematische Annalen, vol. 40 (1891).

§ Loc. cit., p. 270.

velopable, and, if not, what can be said of the other system of asymptotic lines besides that formed by the rectilinear generators? Segre studied questions of each of these types, the algebraic ones in his earlier years, the differential ones later on.* In fact the subject of projective differential geometry was that to which he gave most attention during the later years of his life. He made the acquaintance of Wilczynski at Heidelberg in 1904 and thoroughly appreciated the importance of his work. Yet an outsider cannot help feeling some regret that there still appears a considerable gulf between the Italian doctrine of projective differential geometry, developed by Segre, Bompiani, Fubini and others, and the American contributions of Wilczynski and Gabriel Green, of whom, by the way, no Italian seems to have heard the name.

It is time to turn from Segre's scientific writings to five important articles of an expository character which he produced at various times. The longest, and by far the most important of these was his article in the Mathematical Encyclopedia on the geometry of n-space.† The thought that he must one day complete this, depressed his spirits at times for a good many years, for he was one who took his responsibilities seriously, and he felt in honor bound to put the thing through. Complete it he finally did, thus earning the gratitude of geometers for many years to come. To quote Baker‡

"For completeness of detail, breadth of view, and generous recognition of the work of a host of other writers, this must remain for many years a monument of the comprehensiveness of the man."

In 1891, while still a very young teacher, he wrote for the benefit of his pupils "Su alcuni indirizzi nelle investigazioni geometriche"§ which was reprinted many years later in this Bulletin‖ under the title *On some tendencies in geometrical investigations*. It is a cheering message on the whole, almost as helpful to the young mathematicians of our time and country, as to those of the epoch and place for which it was originally intended. The point which Segre enforces most strongly is that at all costs the beginner must learn to choose really significant subjects. To acquire the instinct to do this, he should study the mathematical classics. He must especially avoid trivial questions and perfectly obvious extensions or generalizations. The young geometer should also be familiar with the leading methods and results of modern analysis, and welcome help from every quarter, regardless of whether his own preference is for synthetic, algebraic, arithmetical or transcendental methods. With regard to rigor, Segre recommends the geometer to be as rigorous as he can, and to own up like a man when he makes use of methods of doubtful parentage.

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* See especially Rendiconti di Palermo, vol. 30 (1910).

† Encyklopädie der mathematischen Wissenschaften, III C 7, pp. 769–972.

‡ Loc. cit., p. 271.

§ Rivista di Matematica, vol. 1 (1891).

‖ Vol. 10 (1903–04).

At the same time he does not consider that geometry is necessarily a succession of δ, ϵ processes, and is not shocked at the judicious use of the convenient phrase "in general".

Very close in spirit to this is his address delivered many years later at the Heidelberg Congress *La geometria d'oggidi e i suoi legami coll' analisi.** His thesis is that geometry and analysis deal with what is essentially the same subject matter, but with different emphasis, terminology and esthetic aim, a view which probably is even more acceptable today than it was when first expressed. He lays special stress on various branches of algebraic geometry, particularly those properties of figures which are unaltered by birational transformation. Near the close, in speaking of the classic memoir of Brill and Noether† he remarks "Tutta una scuola di geometri italiani riconosce nella Memoria di Brill e Noether il suo punto di partenza." This statement falls short of the truth today, through excess of modesty. The present tendency is to take as point of departure two remarkable, and mutually complementary articles which appeared a dozen years before Segre's address, and which deal with the birationally invariant properties of curves.‡ The first by Bertini follows classical algebraic methods, which are set forth with astonishing clearness and elegance. The second by Segre, which shows a remarkable range of mathematical reading from Kronecker to Castelnuovo, follows whenever possible the fundamental methods of the projective geometry of n-space. What has n-space to do with algebraic plane curves? A very great deal. Suppose that we have on a plane curve, not one point, nor a group of points, but a system of such groups, each group containing n points, while the system depends linearly on r independent parameters. These numbers will be birationally invariant. Let us suppose that there are no fixed points common to all of the groups, and that the system is not an involution, i. e., every group containing a general point does not automatically contain certain others. Lastly we may assume that the system is cut by $r+1$ linearly independent adjoint curves $f_0, f_1, \cdots, f_r$. If we write

$$\rho X_i = f_i(x_1, x_2, x_3),$$

we shall have a transformation which carries our plane curve into a space curve of order n in a space of r dimensions, and the groups of our system into the intersections of our new curve with the hyperplanes of its space. For instance, if we have a plane quartic curve with two double points, the conics through these points will meet the curve in a three-parameter system of groups of four points, and will enable us to transform the curve birationally into the elliptic quartic of three-space. This is another aspect of the stereographic projection of a space quartic into a bicircular plane quartic already discussed.

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* Published in volume 2 of the Transactions of that Congress; and also in Rendiconti di Palermo, vol. 19 (1905).

† *Über die algebraischen Funktionen und ihre Anwendung in der Geometrie*, Mathematische Annalen, vol. 7 (1873).

‡ Annali di Matematica, (2), vol. 22 (1893).

We have just used the words "adjoint curves." An adjoint to a given algebraic plane curve is a second which has the multiplicity $s_i - 1$ or more, wherever the original curve has the multiplicity s_i. When the singular points of the original curve are "ordinary", that is to say, when the tangents are all distinct, this definition is perfectly satisfactory. But when the tangents fall together in groups, the singular point may become a very ugly customer, and call for extremely careful treatment in determining the deficiency of the curve and the conditions imposed on adjoints. The problem of determining the number of intersections of two curves absorbed by a common singularity has attracted the attention of geometers from H. J. S. Smith to Enriques. It is the subject of Segre's last long expository paper,* the method of treatment being a careful study of the structure of the resolvent.

Behind Segre the geometer and the expositor, there remained always Segre the teacher and friend. An interesting figure he was in the lecture room. Of medium height and frail, half seated on the end of the table, gesticulating rapidly with his left hand, while his right whirled his watch-chain about with astonishing angular velocity, his whole being was thrown into the task of driving home the essentials of what he had in mind. The subject matter of his course was new every year. There was no limit to the amount of care and patience which he would bestow on one of his pupils. To one who apologized for taking so much of his valuable time with trivial difficulties, he wrote: "Surtout, il ne faut jamais hésiter à me consulter quand vous trouverez une difficulté. Il n'y a que les ignorants et les paresseux qui n'ont jamais de difficultés."

He had a sympathetic, sensitive nature, a spiritual quality, which impressed all who had the privilege of his friendship. Most truly Castelnuovo wrote of him†:

"Corrado Segre ebbe una virtù preziosa concessa a pochissimi eletti: la virtù di rendere migliori tutti coloro che lo avvicinavano. Chi ha potuto vivere qualche tempo nella sua intimità ne ha sentito influenza benefica per tutta la vita."

HARVARD UNIVERSITY

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* Giornale di Matematiche, vol. 36 (1897).

† Loc. cit., p. 359.

OSGOOD ON ANALYTIC FUNCTIONS OF SEVERAL VARIABLES

Lehrbuch der Funktionentheorie. By W. F. Osgood. Part I. Leipzig, B. G. Teubner, 1924. v+242 pp.

It may appear strange to give a review of a part of a volume, an "*Erste Lieferung.*" There is, however, good justification for this. The present 242 pages make a substantial unit by themselves. They comprise the general concepts and theorems which form the basis for a general theory of analytic functions of two or more variables. Subsequent chapters to be added later will be devoted to special branches from the main trunk, such as the theory of $2n$-ply periodic functions of n variables, theta-functions, or automorphic functions in more than one variable; and so on. Since also some considerable time will elapse before the writing and publication of these chapters, it seems both suitable and desirable to review the present brochure.

The need of an up-to-date coordination and systematization of material relating to analytic functions of several variables is almost too apparent to enlarge on. While many of the results and methods for analytic functions of one variable can be carried over without difficulty to functions of many variables, the difficulties at other points are so great, the pitfalls so hidden, the progress so slow, as to make it almost legitimate to say that a profound gulf exists between the theory of analytic functions for a single variable and that for several variables.

The literature pertaining to analytic functions of several variables is rather scattered, and little in the nature of systematic exposition is to be found. Chapter XVII in Goursat's *Cours d'Analyse* and Chapter IX in the second volume of Picard's *Traité d'Analyse* are excellent so far as they go, but are confined to the elementary aspects of the subject. Then there are the Calcutta lectures of Forsyth upon *Functions of Two Complex Variables* (1914), but so different are they in character, scope, and content from the work under review that they increase the desirability and usefulness of the latter. Synoptic reports on analytic functions of many variables are to be found in fourteen pages of the section written by Osgood a quarter-century ago for the Encyklopaedie der Mathematischen Wissenschaften, in Hartogs' Beright in the Jahresbericht der deutschen Mathematiker-Vereinigung (vol. 16 (1907)), and finally in Bieberbach's recent encyclopaedia article (1921), giving the *Neuere Untersuchungen über Funktionen von komplexen Variablen*, of which, however, only a tenth is devoted to analytic functions of several variables. Lastly, I may cite as a forerunner of the present work the Madison Colloquium Lectures of Osgood (1914), treating *Topics in the Theory of Functions of Several Complex Variables.* Naturally, these lectures have also largely the nature of encyclopaedic sketches or skeleton outlines without the connective tissues.

Such a meager amount of literature reveals how strangely little the analytic function-theory has been in fashion mathematically, although its

fundamental importance is undeniable. One of the reasons for this is doubtless its difficulty. The task before Osgood was no easy one: to find out where the known stops and where the unknown begins, to observe carefully the limitation upon the theorems for the case of n variables, to ascertain and weigh the value and significance of the newer methods which have been beaten out where the older methods collapsed. I have often wished for a place in mathematical literature where one could find a comprehensive, connected, and properly restricted modern exposition of analytic functions of several variables, a spot upon which one could plant his feet as on a rock. At last we have it!

Osgood's material is divided into three chapters: Chapter 1. *Integraldarstellungen und mehrfache Reihen. Die erweiterten Räume.* Chapter 2. *Implizite Funktionen. Teilbarkeit.* Chapter 3. *Singuläre Stellen und analytische Fortsetzung. Rationale Funktionen.*

The first chapter, for the most part, contains the theorems which can be obtained by ready extension of the theory for one variable. For this purpose two standard methods are available, the one generalizing Cauchy's integral, and the other using the power series of Weierstrass. Correspondingly, the phrases "Integraldarstellungen" and "Mehrfache Reihen" appear in the title of the chapter. Consistently with volume I, an analytic function $f(z_1, \cdots, z_n)$ is defined as a function which is one-valued at every inner point of a region T and has throughout the region a partial derivative with respect to each of the n variables. The continuity of the function is rightly assumed at the start. In the third chapter, as a much later development, it is shown how this assumption of the continuity of $f(z_1, \cdots, z_n)$ can be removed. The removal was first effected by Osgood when the function is bounded, and later Hartogs succeeded in removing even this restriction.

To treat analytic functions *im Grossen* their behavior at infinity must be considered. Accordingly, it is necessary to explain how the n-fold complex space of the variables z_i or the equivalent real space $(x_1, y_1, \cdots, x_n, y_n)$ of $2n$ dimensions is to be extended by the invention of points and configurations at infinity ("Die erweiterten Räume"). One of the admirable features of the book is the treatment of this matter. The reader may recall how Bôcher (this Bulletin, vol. 20 (1914), p. 185) has emphasized that there were other ways than the conventional one of extending space and that the object in view must determine the character of the space-completion "at infinity." Osgood re-emphasizes this view by indicating several possible extensions for the space of the n variables of our analytic functions. The usual extension gives what the author fittingly calls the "space of analysis," and is obtained by adding a single point in each of the n complex planes of our n variables $z_1, \cdots, z_n$. In consequence, the *infinite region* consists of all points $(z_1, \cdots, z_n)$ for which one or more of the n coordinates is infinite. Another kind of space, which I shall refer to as *complex projective space*, results by basing the extension upon the familiar linear projective group in n variables (allowed to be complex), and this has been fruitfully employed by Osgood himself.

The second chapter is rooted in Weierstrass's so-called preparation theorem, with which it opens. Upon this as a foundation is built the theory of implicit analytic functions, and another application is to the concept and theory of reducibility (Teilbarkeit). The former of these two topics is treated with great fullness, as we would wish, since the field of implicit functions has been especially studied in America by Osgood and his former pupils, Dederick, Clements, etc., and also by Bliss. Systems of simultaneous analytic equations and parameter representation *im Kleinen* are also considered.

In the third chapter the subject of singular points and configurations is first introduced, and it is devoted in large part to an exposition of the theorems of Hartogs, Levi, Cousin, etc., which treat of analytic behavior, meromorphic behavior, etc., *im Grossen*. It has the special interest of bringing to the reader the modern methods devised to establish recalcitrant theorems which, though perhaps long surmised, have occasioned great difficulty and ofttimes have been demonstrated either restrictedly or inadequately.

This division of contents is very natural, and altogether logical. But there is so much material to pack away as to give trouble in sectional subdivision—at least, it seems so to the reviewer. One finds a certain lack of swing and natural evolution, particularly in the first two chapters. This seems to me to be the one and only marked deficiency of the book. To apply a trite phase, the trees in the wood are so thick that the map is easily lost. The reader will, I think, find it helpful to keep on hand Osgood's Colloquium Lectures, and use it as an aid in getting a bird's eye view over the whole. No one, however, can fail to feel the difficulty of organizing so much material nor be unaware that the parts fit together, like the stones of a grand mosaic, to make one great design.

The richness of the material is a constant delight. It is impossible to read without perceiving the tremendous amount of study which must have been spent in collating and correlating, and without prizing the thoroughness and completeness with which this has been done. Further, the character of the analytic function theory for two or more variables is such as to demand scrupulous care, and we are glad that the task has fallen to an Osgood. We congratulate ourselves that he did not conventionally stop with the analytic theory of a single variable.

The successive topics are developed very fully. In Chapter I there is frequently a rapid change from one topic to another having little relation to it. To cite only a single instance, the section on homogeneous coordinates is succeeded by one on Laurent's series, and this is followed by one on analytic continuation. The section on Laurent's series is thus sawed off from other work on series. The matter is, however, very closely interwoven. If one attempts to select and study some particular topic here or there, he will usually find it imperative to go back and digest something that has gone before, and from this he may have to turn to something still further back. There is also continual reference by chapter and section numbers to volume I. Osgood presupposes justly that the reader should already have a firm grasp on the function theory for one

variable. The book is not one to be used to get an understanding of function theory for one variable from that for n variables by putting $n=1$. One must be content to proceed slowly and to think and ponder as he proceeds. When he has so done, he will be well satisfied with the grip obtained.

I turn now to a more detailed notice of the features of Chapter I. After a variety of topics, such as the Differential, the Mittelwerthsatz, Liouville's Theorem, and certain work on implicit functions, he paves the way for the all important Cauchy integral theorems. In §8 we find the needed theorems regarding the analytic character of the integral

$$\int_{C_1} dt_1 \int_{C_2} dt_2 \cdots \int_{C_m} f(t_1, t_2, \cdots, t_m; z_1, \cdots, z_n)\, dt_m,$$

where f is continuous in its $n+m$ arguments for values of the t_i on "regular" curves C_i in the t_i-planes and for values of the z_i in n regions of the complex plane of these n variables, and furthermore is analytic in the z_i for these values of the arguments. In §9 the form of the region is specified to which the work is subsequently largely confined. This is what he terms a *cylindrical* region (Zylinderbereich) in the space S_{2n} of $2n$ dimensions which is obtained by associating arbitrarily the values $z_i = x_i + iy_i$ belonging to the inner points of n independent regions (not necessarily simply connected) in the n-planes of the separate variables. The care of the author in specializing the regions he uses and thus preventing his reader from mentally extending conclusions to any old kind of region is a laudable characteristic of the book. After the introduction of Cauchy's integral formula for $f(z_1, \cdots, z_n)$ and the deduction from it of various important consequences, the scene shifts to a rather uninteresting analogue of the mean value theorem of the differential calculus and then to a nineteen page discussion of "Mehrfache unendliche Reihen," which by the definition are absolutely convergent. For power series $P(z_1-a_1, \cdots, z_n-a_n)$, it is noted that the region of absolute convergence may consist not merely of the simply connected region of $2n$ dimensions usually considered, but also, possibly, of regions of $2k$ dimensions ($k<n$) in the coordinate planes. I mention this casually to illustrate thereby the variety of interesting and minor features which arise from the author's care. The sections on series culminate, of course, in the familiar Cauchy-Taylor power expansions in n variables. In the sections next following, the space for the analytic functions is extended appropriately to infinity, first without the introduction of homogeneous coordinates. Then, after their introduction, he goes on to establish that in the resulting $2(n+1)$-fold space of the new homogeneous variables corresponding to the space S_{2n} of the original variables z_i, the function will be analytic or meromorphic in terms of the new $n+1$ homogeneous variables if it is so in the corresponding space of the original n variables. We are next surprised by the remarkable theorem (p. 57) that if $f(w,z)$ is analytic and finite for the region $|w|+|z|>G$, where G denotes a fixed number, then f is a constant. This theorem is one of few which is without an analogue for the analytic function of one variable. The subject of analytic continuation of functions by successive overlapping regions is, of course, conventional, but in considering analytic extension along a regular curve in

S_{2n} the author takes the pains to introduce a specific form of region both for extended and unextended space to take the place of the circle of convergence used in analytic extension along a curve in the complex plane of a single variable, and this is done both for the space of analysis and for complex projective space.

The first three sections of Chapter II relating to the so-called preparation theorem of Weierstrass are particularly good, just as we would wish at so important a place. For the sake of what is attached to it, I repeat the theorem here: If an analytic function $F(w, z_1, \cdots, z_n)$ vanishes at a point $(b, a_1, \cdots, a_n)$—for convenience taken to be $(0, a_1, \cdots, a_n)$—but without vanishing identically around it, then in an appropriate vicinity, $|w|<r$, $|z_j-a_j|<h$, $(j=1, \cdots, n)$, we can express F in the form

$$F(w,z_1, \cdots, z_n) = (w^m + A_1w^{m-1} + \cdots + A_m)\,\Omega\,(w, z_1, \cdots, z_n),$$

where the A_k are functions of $z_1, \cdots, z_n$, which are analytic and vanish at the point $(a_1, \cdots, a_n)$, while Ω is a function which does not vanish in the vicinity specified. The case in which $F(w, a_1, \cdots, a_n)$ vanishes identically is first discharged in the manner of Weierstrass by an appropriate linear transformation in the $n+1$ variables $w, z_1, \cdots, z_n$, and then Osgood goes on to prove that the form above given to the preparation theorem will hold also in that case for $n=1$, if we insert before w^m in the above formula a coefficient $A_0(z)$ which vanishes at $z=0$. By means of a special example Osgood shows further that a like form will not hold for $n>1$. This failure to hold is just one of many instances showing how extremely careful one must be in generalizing the ordinary analytic theory for one independent variable.

The first factor in the above preparation theorem brings before the reader an instance of what the author calls a pseudo-polynomial, that is, an expression $a_0w^m+a_1w^{m-1}+\cdots+a_m$ in which the a_k denote functions $a_k(z_1, \cdots, z_n)$ which are all analytic at some point $(c_1, \cdots, c_n)$ under consideration, called the *Spitze* of the pseudo-polynomial. The existence of the pseudo-polynomial in the above preparation theorem suggests the possibility of further decomposition of the analytic function into pseudo-polynomial factors. We are thus led to the theories of divisibility, reducibility, resolution into prime factors, etc., which follow. It may be remarked that the theories are almost altogether "im Kleinen." By equating the pseudo-polynomial to zero we are led to implicit function theory and also to a study of functions on the resulting pseudo-algebraic "Gebilde." A considerable part of this second chapter is rather specialized work on implicit function theory, the Jacobian, Riemannsche Räume generalizing Riemannian surfaces, systems of pseudo-algebraic equations, etc. For a first reading, one can well pass over a large portion of this chapter, confining his attention after §10 to the parts relating to functions on a pseudo-algebraic Gebilde and to parameter representation, and proceed directly to Chapter III. Indeed, we rather regret that some of the material was not postponed to a Chapter IV, so that first essentials could be more rapidly reached and picked out.

Chapter III compels one's interest. In the opening section, the author first discusses the topic of the singularity of the analytic function. The

chapter deals chiefly with the unessential singular point (ausserwesentliche singuläre Stelle), in the vicinity of which the function can be expressed in the form $G(z_1, \cdots, z_n)/H(z_1, \cdots, z_n)$, where G and H are both analytic and the latter vanishes at the point. When the numerator does not vanish, we get an unessential singularity of the first kind, which is the analogue of the familiar pole, and when it vanishes, an unessential singularity of the second kind, which much resembles the essential singularity of an analytic function of the single variable. When all the singularities in a given region are unessential, the function is, of course, called meromorphic.

Some of the greatest difficulties in the study of analytic functions of many variables arise from the fact that the zeros and singularities are never isolated, but form a continuous configuration. Thus, unessential singularities of the first kind fill the $(2n-2)$-fold manifold $H(z_1, \cdots, z_n) = 0$, and those of the second kind (in whose vicinities the function is unbounded) fill a $(2n-4)$-fold manifold. Not only are these (obviously) excluded in the two remarkable theorems due to Kistler, which the author goes to prove, but also all other singularities. The first of these theorems is as follows. If a function $F(z_1, \cdots, z_n)$ is analytic in a $2n$-dimensional region T except possibly for the points of a $(2n-2)$ fold manifold and is furthermore limited, then the function can be continued analytically over the *entire* region T, so that the points of the exceptional manifold, if singular, are removable discontinuities. The second theorem is like the first except that the possible exceptions make at most a $(2n-4)$-fold manifold, while the restriction that the function shall be limited is altogether omitted. The method of proof of these theorems and of others consists mainly of adroit use of Cauchy's integral formula. The same tool gives a number of very interesting theorems, due for the most part to Hartogs, which continue a function analytically beyond the region of its analytic definition. These culminate in Hartogs' fine result that a function $f(z_1, \cdots, z_n)$ which is one-valued and analytic on the boundary of a "regular" cylindrical region permits of analytic extension over the whole interior of the region. This conclusion is noteworthy inasmuch as the corresponding statement is not true in the case of a function of a single variable. After the theorems of Hartogs come the noteworthy theorem of E. E. Levi relating to meromorphic continuation of an analytic function. In particular, the theorem of Hartogs last cited will continue to hold if the word "analytic" is replaced by "meromorphic."

These results can only hint at the wealth of interesting theorems in this third chapter of Osgood's work. We find in accessible form Cousin's extensions of the well known theorems of Weierstrass for the representation of an entire function as a product of factors exhibiting zeros, and of Mittag-Leffler for the representation of a meromorphic function with arbitrarily distributed poles. The extension of the former given by Cousin contained a vitiating error, in that Cousin did not properly delimit the connectivity of the region. Gronwall, who started on the consideration of analytic functions of several variables in his Uppsala thesis, later rendered an

important service in detecting the hidden error and properly restricting the result.

Finally, we reach the theorem that a function meromorphic in a simply connected region (which may be the entire finite space) can be expressed as the quotient of two functions analytic over the region. This is established by Osgood without the use of the harmonic functions of many variables employed by Poincaré in the first demonstration given, the theory of which is even today without systematic development. Then follows the Weierstrass-Hurwitz theorem that a function meromorphic over the entire plane is a rational function. Special attention may be called to the remarkable fact that the theorem is not merely valid for "the space of analysis", but was proved by Osgood to hold also for complex projective space (and even for space extended in a mixed manner to infinity (Jackson)). I can not forbear to mention the simple proof that every one-to-one analytic* transformation of the extended space of analysis into itself has the form

$$z_i = \frac{a_k z_k + b_k}{c_k z_k + d_k}, \qquad (i = 1, \cdots, n),$$

where the n values of k appearing in the right-hand members are the numbers $1, 2, \cdots, n$ in arbitrary order. This theorem is well known for the case of a single variable z, though even for this case the reader may experience trouble in ascertaining where a demonstration is to be found. References to literature are abundantly given by Osgood, but at this point, as at some others, we find none at all. We have the suspicion, therefore, that the general proof here given is the authors' own, as well as a corresponding theorem for complex projective space.

The volume is by no means easy to read, but grows simple if one delves and ponders. It shows the same thorough, massive scholarship, extending to all details, which I commented upon so favorably in my review of volume I. If we deplore again that the book could not have been written in English, we recognize a certain fitness that it appears in German, for it displays the inspiration of a German graduate training, as well as the persistent, accurate scholarship so characteristic of German work.

The author wishes me to call attention to the following corrections. Page 129, line 4 from the top. Reference should be to the Beispiel von §2 (not §22). Also in the middle of the page read $2(y/x)$ in place of $2(y/z)$. Page 33, line 3 from bottom. Instead of $a_k \neq 0$, $k = 1, \cdots, n$, read $\sum |a_k| > 0$, $(k = 1, \cdots, n)$.

E. B. VAN VLECK

* Of course, with proper modification in the infinite region.

THE NEW RIEMANN-WEBER

Die Differential- und Integralgleichungen der Mechanik und Physik, the 7th Edition of Riemann-Weber's *Partiellen Differentialgleichungen der mathematischen Physik*, edited by P. Frank and R. von Mises. *Erster, mathematischer Teil*, edited by R. von Mises with the assistance of L. Bieberbach, C. Carathéodory, R. Courant, K. Loewner, H. Rademacher, E. Rothe, C. Szegö. Braunschweig, Vieweg, 1925. 687 pp.+76 figures. Paper, 40 Rentenmark; bound, 44 Rentenmark.

As a compliment to the memory of Riemann and H. Weber, the present work is nominally issued as the seventh edition of Riemann-Weber, *The Partial Differential Equations of Mathematical Physics*. The fourth edition of Riemann's treatise under the above title was Heinrich Weber's first contribution. The fifth edition (Weber's second) was issued in 1910 and the sixth edition was a mere reprint of the fifth. H. Weber before his death had planned an edition which should be entirely rewritten with the cooperation of a group of younger mathematicians and physicists. His chief collaborators were to be his son, R. Weber, and J. Wellstein. The latter two did not survive the elder Weber very long, and the work was then undertaken by R. von Mises with the cooperation of the seven listed above.

In the main the present plan differs from the original in a division into a mathematical part (the present volume under review), and a second, a physical part announced for an early appearance. The second part is to be edited by P. Frank and written with the help of R. Furth, A. Sommerfeld, F. Noether, Th. v. Karman, and E. Trefftz. Though written by eight collaborators, the present work exhibits a degree of unity far beyond what might be expected. To be sure, the field as a whole lends itself by its very nature to systematic treatment and unification. But the skill with which this is accomplished is due chiefly to the labor of the editor and chief contributor, von Mises.

The following is a summary of the table of contents by chapters.

Part I. Fundamental preliminary theory. 1. Real functions. 2. Linear forms. 3. Complex variables. 4. Infinite series and products. 5. Calculus of variations.

Part II. Ordinary differential equations. 1. Problems involving initial conditions. 2. Boundary value problems of the second order. 3. Particular functions corresponding to second order boundary value problems. 4. Development of arbitrary functions. 5. Developments in Bessel's functions.

Part III. Integral equations and potential. 1. Summary of problems and results. 2. Solution of integral equations. 3. Applications of integral equations to boundary value problems. 4. Potential.

Part IV. Partial differential equations. 1. Problems involving initial values. 2. Potential in the plane. 3. Potential in three-space. 4. Boundary value problems for partial differential equations of the second order.

5. Other problems in partial differential equations. 6. Calculus of variations and boundary value problems.

There is some overlapping in the various chapters, but this is of a helpful kind. Beginning with the fundamentals of the theory of functions of a real variable and of a complex variable, through the calculus of variations and vector and tensor analysis, the work proceeds with a definite end in view. All of these matters are to join on to specific topics in mathematical physics.

It is clear to the reviewer that the present plan purposes not merely to rearrange material but to offer a well organized presentation of the mathematical "tool-box" of modern theoretical physics. That the reader is carried well beyond undergraduate courses in calculus and differential equations is attested by such topics as "functions of limited variation," "Stieltjes and Lebesgue integrals," "conformal mapping," "extremals," "canonical coordinates," "contact transformations," "characteristic functions," "Dirichlet's principle," "infinitely many variables," etc. The volume is indeed of value alike to the worker in physics and to the mathematical specialist, even though not all the recent advances in the various fields, especially results of a very particular nature, are included. But in the main, where the method is of importance, a detailed discussion is at hand. For example, for special results, the present volume is not as complete as a book which appeared a year earlier (1924) written by one of the present writers, R. Courant, *Methoden der Mathematischen Physik*, which forms volume XII of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. The present book, however, offers a wider perspective. The treatment is not encyclopedic, nor exhaustive, but it lays a broad foundation and provides a sufficient framework around which courses in "pure" or "physical" mathematics may be built.

To give a detailed account of the contents of the book would be to rehearse not only the recent developments of the past thirty years work which may be included under the heading of differential and integral equations, but most of the modern and classical developments of analysis. The names of von Mises, Bieberbach, Carathéodory and Courant, well known both as research workers and excellent expositors, as well as the others, are the best recommendation one can give for the value of the present work. Supplementary references to more extensive and special treatises are to be found at the end of most of the chapters, together with occasional mention of original memoirs.

H. J. ETTLINGER

SHORTER NOTICES

Mécanique Analytique et Théorie des Quanta. By G. Juvet. Paris, Blanchard, 1926. 6+153 pp.

Two-thirds of this book deal with general mechanical principles and perturbation theory. The opening chapters are concerned mainly with the Hamiltonian canonical equations and the Hamilton-Jacobi partial differential equation. The style and method of demonstration are usually varied enough from those given in the standard texts on dynamics so that even the discussions and proofs of the standard theorems do not seem hackneyed, but occasionally one feels that the mathematical framework is a bit more elaborate than necessary for understanding most of the literature of the quantum theory. A happy feature is the emphasis that the canonical equations furnish the "characteristic curves" of the Hamilton-Jacobi partial differential equation, for this observation to a certain degree bridges the gap that usually exists between the physicist's presentation of the Hamiltonian dynamical technique and the analysis of general first-order partial differential equations found in such treatises as those of Goursat, Jordan, etc. In Chapter V the separation of variables is considered at some length. The two following chapters are devoted to perturbation theory and the methods of successive approximation developed by Delaunay, Lindstedt, and Bohlin. No attempt is made to delve into the intricate question of the convergence of the trigonometric series. One admires the candor with which Juvet acknowledges that his treatment in Chapter VII follows closely that in Poincaré's *Méthodes Nouvelles de la Mécanique Céleste.* One might wish that all authors were equally frank, for to quote Juvet, "Nous ne croyons pas qu'il suffice de citer un auteur lorsqu'on lui prend tout ce qu'on dit, il convient de dire que tout lui revient." Nevertheless this chapter is of distinct value because it summarizes and brings together material which is rather widely scattered through the various chapters of volume II of Poincaré's treatise. It is indeed interesting that the perturbation technique of the astronomer has lent itself with so little modification to the Bohr theory of atomic structure.

The last third of the book deals with application of dynamics to the quantum theory of atomic structure. The analysis is, of course, entirely on the basis of the old Bohr theory, as it was written before advent of the new matrix dynamics of Born, Heisenberg, and Jordan, or of the wave mechanics of Schrödinger. Chapter XII presents Burgers' first method of proving the principle of adiabatic invariance, and no mention is made of his second method (summarized in an appendix of Sommerfeld's *Atombau*), which in our opinion is simpler once the transformation theory of dynamics has been established. The last few pages contain an interesting discussion of the relative utility of the various perturbation methods developed in Chapters VI and VII, but these methods are not illustrated

by applications to specific problems such as the anharmonic oscillator, helium atom, or dispersion. The quantum theory is replete with problems of this type which can be handled only by successive approximations, and it seems unfortunate that no space could be found for their inclusion after such careful development of the general perturbation theory. In fact, not even the appropriate references to the literature are given. The separation of variables, however, is illustrated by the conventional examples of the Stark effect and relativity corrections in hydrogen.

Professor Juvet's book covers a field quite distinct from most other treatises, and is written with characteristic French smoothness of style. The general flavor is mathematical rather than physical, for, to quote from the preface, "The aim is simply to put a mathematical instrument at the disposal of those who are interested in physics, and at the same time to acquaint mathematicians with one of the most elegant applications of analytical mechanics."

J. H. Van Vleck

The Quantum Theory of the Atom. By George Birtwistle. Cambridge University Press, 1926. 236 pp.

Just what form the quantum theory of the atom will take by the time this review appears in print it is impossible to say; but whatever the quantum theory may become, its original form will always be a matter of interest. This theory assumes a definite model of the atom—the Rutherford atom—and applies to this the laws of classical dynamics. From the continuous manifold of dynamically possible orbits, the quantum theory allows only a discreet manifold.

Mr. Birtwistle has written an excellent account of the quantum theory based on these ideas. He first shows how the quantum theory was introduced by Planck in obtaining his formula for the distribution of radiant energy; he then describes Bohr's original theory of the hydrogen spectrum, Einstein's photo-electric equation and his deduction of Planck's formula. Then follow a number of chapters on the general dynamical theory, which is needed in the applications of the quantum theory that follow: to the fine structure of spectral lines, to the Stark and the Zeeman effects. A brief account is given of spectral series together with Bohr's theory of the building up of atoms and the theory of band spectra. The classical theory of perturbations of dynamical systems is next given, with an application to the anharmonic oscillator. The last chapter contains a sketch of the dispersion theory of Kramers.

This book was published while the quantum mechanics of Heisenberg was in process of development, and this theory is just mentioned at the end of the volume. At the present time the undulatory mechanics of de Broglie and Schrödinger is attracting much attention. But whatever may be the fate of these recent speculations, there can be no question as to the value of the older form of the quantum theory in view of its achievements; and it would be difficult to find a better account of it than in this book.

E. P. Adams

Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle. By Serge Bernstein. Paris, Gauthier-Villars, 1926. 10+207 pp.

This book, which is one of the Borel series of monographs, forms an exposition of the course of lectures given by Professor Bernstein at the Sorbonne in May, 1923. The results are for the most part due to the author himself, and the majority of them appear here in print for the first time. The work as a whole is a development of the methods of Tchebycheff in the study of the field indicated by the title. The great variety of interesting and important results that are here contained in some two hundred pages, furnishes the best possible support for the writer's contention that the ideas of Tchebycheff are of fundamental importance in this domain of the theory of analytic functions of a real variable.

The first chapter is concerned with extremal properties on a finite segment of polynomials and other functions depending on a given number of parameters, and develops the algebraic basis for the whole subject. It contains among other important theorems a classical result of the author concerning the best approximation to $|x|$ by polynomials of given degree. We also find here his fundamental theorem with regard to the relationship between the maximum modulus of a polynomial and that of its derivative.

In the second chapter a study is made of extremal properties on the entire axis of reals of algebraic functions and of certain types of integral functions. In connection with the application of these properties to polynomial approximation there is introduced the notion of a "function of comparison", $\phi(x)$, and the extremal properties of the expression

$$\epsilon_n = \left| \frac{f(x) - P_n(x)}{\phi(x)} \right|,$$

where f is a given function and P_n an arbitrary polynomial of degree n, are studied. It will be seen that this use of a function of comparison is analogous to the notion of relatively uniform convergence introduced by E. H. Moore. We find also in this chapter the interesting theorem that any continuous function of a real variable, $f(x)$, such that $\lim_{x \to +\infty} f(x) = A$, can be uniformly approached on the whole axis of reals by means of rational fractions possessing given poles $\alpha_n \pm \beta_n$, provided the series $\Sigma |\beta_n| / (\alpha_n^2 + \beta_n^2)$ diverges. On the other hand, the uniform approach to $f(x)$ by rational fractions, possessing poles such that the above series is convergent, implies the fact that $f(x)$ is an analytic function having no singularities in the finite plane other than poles at $\alpha_n \pm \beta_n$. In the latter part of the chapter the fundamental results regarding the relationship between the maximum of the modulus of a polynomial and that of its derivative, referred to in connection with Chapter II, are generalized to the case of integral functions.

The third chapter deals with the question of the best approximation to analytic functions possessing given singularities. As that topic had previously been treated by de la Vallée Poussin in his *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, also of the Borel series, Pro-

fessor Bernstein places the emphasis on those problems that were not discussed in the above mentioned monograph. He studies in particular the determination of the successive terms in the asymptotic development of the best approximation and the problem of the best approximation to a function having an essentially singular point. This latter is a problem of unusual difficulty, and it is noteworthy that in a wide variety of cases a solution of the problem can be obtained by the use of the same methods as in the case of polar singularities.

The book closes with two notes. The first one, which is rather extensive, develops certain interesting relationships between the problem of analytic extension of analytic functions of real variables and that of polynomial approximation to such functions. The second note, which is brief, deals with a property of analytic functions of genre zero.

As a whole the book has to a very great extent that admirable characteristic of the Borel series: namely, that its interest lies not only in the topics specifically treated, but also in the variety of problems for further research that are either definitely indicated or incidentally suggested.

C. N. Moore

Potentialtheorie. By Dr. W. Sternberg. Two volumes. I. *Die Elemente der Potentialtheorie*, 136 pages, 1925. II. *Die Randwertaufgaben der Potentialtheorie*, 133 pages, 1926. Volumes 901 and 944 of the Sammlung Göschen. Berlin, Walter de Gruyter and Co.

The two volumes give a brief introduction to the theory of the various potentials in two and three dimensional space and to the two boundary value problems. The theorems and methods are in general well known. The author has attempted to reduce the necessary preparation of the reader to a minimum. A knowledge of the calculus is sufficient except in the case of the chapter on the Fredholm equation. After a short *mengentheoretische* introduction, the first volume takes up the definition and some properties of volume, surface, and double layer potentials. It is shown that the definition of the potential as a solution of Laplace's equation is coextensive with the usual integral definition. This new definition necessitates a new definition of mass and in this the author follows Plemelj. After a chapter on the necessary variations of Stokes' Theorem, the volume closes with a discussion of the continuity properties of the potentials and their derivatives. In the first part of the second volume the author confines himself chiefly to a statement of the Dirichlet and Neumann problems, their solution for the circle and sphere and a discussion of the Green's function. No mention is made of the method of Neumann or of the method of *balayage* of Poincaré. The remainder of the volume is given over to the theory of Fredholm's equation and its application to the boundary value problems. The chapter on the Fredholm equation is rather too brief to be useful to one who knows nothing of the theory. The work is carefully written and though very concise, is readable and is to be recommended.

C. A. Shook

Die Grundlagen der ägyptischen Bruchrechnung. By DR. O. NEUGEBAUER. Berlin, Springer, 1926. 45 pp., and six tables.

This monograph is on the philosophy of Egyptian mathematics. The author advances the thesis that "Egyptian mathematics rests exclusively upon the fundamental operation of addition," that it is wrong to attribute to the Egyptians a knowledge of the four fundamental operations of arithmetic. According to him, Eisenlohr, Peet, Cantor, Hultsch and others have read into Egyptian mathematics processes which are not really there. It seems to the reviewer that Neugebauer establishes the possibility of explaining the arithmetical operations of the Egyptians on the additive principle alone, but that he has not demonstrated that the Egyptians actually limited themselves to that principle. In general, it seems improbable that Egyptian mathematicians able to compute correctly the volume of the frustrum of a pyramid, operated with an arithmetic that did not involve the idea of multiplication. However, Neugebauer's claim is worthy of careful study. An interesting process in Egyptian mathematics, emphasized by our author, is the equivalent of the modern so-called Russian process of multiplication. To multiply 7 by 5, put down

/1	7
2	14
/4	28

and add the numbers marked by a stroke.

FLORIAN CAJORI

Mathematische Instrumente. By F. A. Willers. Sammlung Göschen. Berlin und Leipzig, Walter de Gruyter, 1926. 144 pp.

This little monograph, written by Dr. Willers of the Technische Hochschule of Charlottenburg, is No. 922 of the well known Göschen collection, and contains a clear and adequate description of the most important and useful mathematical instruments and machines.

In the introduction, Dr. Willers stresses the necessity of familiarizing oneself with the mechanical construction and with the theory of such instruments, in order to be able to handle them intelligently and rapidly. Only in this manner may one gain an enormous advantage over the drudgery of many mental calculations.

The booklet is written especially with this purpose in view, and is divided into eight chapters, dealing with continuous calculating apparatus, like slide rules, calculating machines, instruments for graphing and measuring curves, planimeters, harmonic analysers, mechanical solution of boundary problems, geometrical apparatus, like affinographs, pantographs, and perspectivographs.

Altogether, Dr. Willers' little treatise will be a welcome addition to the literature for all those who do not want to make a special study of mathematical instruments but who wish to be able to use such auxiliaries, when necessary or desirable, for the solution of some particular problem requiring arduous numerical work.

ARNOLD EMCH

Analyse Fonctionnelle. By Paul Lévy. Mémorial des Sciences Mathématiques, fasc. V. Paris, Gauthier-Villars, 1925. 56 pp.

The author gives in this brief discussion an exposition of what is necessarily only a part of the subject of functional analysis, but in such a way as to impress the reader with the existence of new ideas under the sun. The condensation naturally enforces slow reading.

Chapter I gives the fundamental notions of function space and the generalized definition of distance, in particular the space Ω, where the square of the distance between two functions is the integral of the square of their difference, and the space ω, in which distance may be defined as the maximum of the numerical value of the difference. The linear functional in the two spaces takes on different forms, in ω being given by a Stieltjes integral and in Ω by a Lebesgue integral; in Ω, orthogonal axes may be regarded as defined by a set of an infinite number of orthogonal functions $f_n(x)$; and a function $\Sigma a_n f_n(x)$ as having coordinates $a_1, \cdots, a_n, \cdots$. Consideration of the vectors in this space yields various trigonometric formulas. Consideration of the linear transformation in the two spaces leads to different types of integral equations.

Chapter II gives an introduction to equations in functional derivatives. The reader will probably turn to references where he can find more detail, but he will be able here to get a notion of complete integrability and of characteristics.

In Chapter III is developed the idea of integration in function space. On account of the fact that the ratio of the volumes of two similar figures becomes infinite or zero as the number of dimensions becomes infinite, the idea of volume must be replaced by that of average. Here again, however, we meet the same difficulty, in that the neighborhood of one position is apt to predominate to an infinite extent in the determination of the average. (This aspect of infinity will be more or less familiar to the reader as the characteristic of one of the well known methods of arriving at Maxwell's law of distribution for the velocities in a gas of a large number of molecules.) It becomes desirable therefore to consider other kinds of weighted means.

The author flirts throughout in engaging fashion with applications to harmonic functions, considering $\partial^2 g(M_1, M_2)/\partial n_1 \partial n_2$ as a function of curves, the equation in functional derivatives of the Dirichlet problem and the nature of harmonic functions in space of an infinite number of dimensions.

The theories expounded owe, of course, to the founders of the subject, and much also to Lévy himself and to Gateaux, Wiener, Fréchet and Daniell.

G. C. Evans

NOTES

The opening number of volume 29 of the Transactions of this Society (January, 1927) contains the following papers: *Sets of independent postulates for the arithmetic mean, the geometric mean, the harmonic mean, and the root-mean-square*, by E. V. Huntington; *Irregular differential systems of order two and the related expansion problems*, by M. H. Stone; *Some problems in the theory of interpolation by Sturm-Liouville functions*, by C. M. Jensen; *Singular ruled surfaces in space of five dimensions*, by E. B. Stouffer; *On a type of completeness characterizing the general laws for separation of point-pairs*, by C. H. Langford; *Integers and basis of a number field*, by N. R. Wilson; *Implicit functions and their differentials in general analysis*, by T. H. Hildebrandt and L. M. Graves; *Application of the theory of relative cyclic fields to both cases of Fermat's last theorem* (second paper), by H. S. Vandiver; *Riemann integration and Taylor's theorem in general analysis*, by L. M. Graves; *Alternatives to Zermelo's assumption*, by A. Church; *On a general theorem concerning the distribution of the residues and non-residues of powers*, by J. M. Vinogradov; *On the bound of the least non-residue of nth powers*, by J. M. Vinogradov; *On convergence factors in multiple series*, by C. N. Moore.

The Executive Committee of the coming International Congress of Mathematicians, of which Professor S. Pincherle is chairman, has selected the dates September 1–10, 1928, for this meeting. American mathematicians will be gratified to know that these particular dates were selected in order to make possible attendance from America. A preliminary announcement will be issued in the near future.

At the Annual Meeting to be held in conjunction with the meetings of the American Association for the Advancement of Science at Nashville, Tennessee, December 27–28, 1927, Professor James Pierpont will represent this Society at the usual joint session in which several organizations participate; he will deliver an address on *mathematical rigor, past and present*.

The office of the Society in New York has been moved to Room 1324 of the newly erected Physics Building of Columbia University. Mail addressed to the old street number will continue to reach the office properly; but telegrams and express packages should be addressed to the new office.

At its annual meeting of 1926, the American Society of Mechanical Engineers considered the question of organizing a professional division of mechanics, physics, and applied mathematics. A committee consisting of S. Timoshenko (chairman), A. L. Kimball, and H. A. S. Howarth was appointed to formulate a definite plan for this division.

The Ackermann-Teubner memorial prize for 1926 has been awarded to Professor Wilhelm Blaschke, of the University of Hamburg, for his book *Kreis und Kugel* (Leipzig, Veit, 1916).

The Ernst Abbe memorial prize for physics has been awarded to Professor Wilhelm Wien, of the University of Munich.

The Italian Society of Sciences (the XL) has awarded its mathematics prize for 1926 to Professor A. Comessatti, of the University of Padua, for his researches in algebraic geometry.

The gold medal of the Royal Astronomical Society has been awarded to Professor Frank Schlesinger, of Yale University, for his work in stellar parallax and astronomical photography.

Dr. Elihu Thomson, of the General Electric Company, has been awarded the Faraday medal of the British Institution of Electrical Engineers for 1927.

The Astronomical Society of the Pacific has awarded its Bruce gold medal for "distinguished services to astronomy" to Professor H. H. Turner, of Oxford University.

The Collingwood prize of the American Society of Civil Engineers has been awarded to C. V. von Abo, of Johannesburg, South Africa, for his paper on *Secondary stresses in bridges*.

Dr. C. E. Guye, professor of physics at the University of Geneva, has been elected a corresponding member of the Paris Academy of Sciences.

Professor Max Planck, of the University of Berlin, has been made an honorary doctor of engineering of the Berlin Technical School.

At the University of Paris, Dr. Eugène Bloch, maître de conférences, has been promoted to the chair of theoretical and celestial physics.

Dr. P. Cousin, professor of the differential and integral calculus at the University of Bordeaux, has been made dean of the University.

Professor F. Cecioni, of the University of Catania, has been transferred to the University of Pisa. He will teach algebraic analysis.

Professor O. Chisini, of the University of Milan, has been promoted to a full professorship of analytic geometry.

Dr. E. Fermi has been appointed to an associate professorship of theoretical physics at the University of Rome.

Dr. E. Persico and Dr. G. Sansone have been appointed to associate professorships at the University of Florence. The former will teach theoretical physics and the latter the calculus.

Dr. R. Serini has been appointed to an associate professorship at the University of Pavia. He will teach theoretical physics.

The following 44 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1926; the university, month in which the degree was conferred,

minor subject (other than mathematics), and title of dissertation are given in each case if available.

Evelyn F. Aylesworth, California, May, *The dielectric constant of atomic hydrogen and the Stark effect.*

Wealthy Babcock, Kansas, June, physics, *On the geometry associated with certain determinants with linear elements.*

Mr. H. W. Bailey, Illinois, May, astronomy, *The summability of single and multiple Fourier series.*

R. W. Barnard, Chicago, December, *The Fredholm theory of linear integral equations in general analysis for quaternionic-valued functions.*

Martha H. Barton, Johns Hopkins, June, Physics, *Some applications of the generalized Kronecker symbol.*

Florence Black, Kansas, June, physics, *A reduced system of differential equations for the invariants of ternary forms.*

H. L. Black, Illinois, May, physics, *A Cremona group isomorphic with the group of the twenty-seven lines on a cubic surface.*

E. T. Browne, Chicago, September, *Involutions that belong to a linear class.*

N. B. Conkwright, Illinois, May, astronomy, *The summability of Birkhoff series.*

A. E. Cooper, Chicago, June, *A topical history of the theory of quadratic residues.*

A. H. Copeland, Harvard, *Studies on the gyroscope.*

C. M. Cramlet, Washington, June, physics and astronomy, *Invariant tensors and their application to the study of determinants and allied tensor functions.*

H. T. Davis, Wisconsin, June, physics, *An existence theorem for the characteristic numbers of a certain boundary value problem.*

M. S. Demos, Harvard, *The group characteristics of the general and special quaternary linear homogeneous congruence groups.*

R. D. Doner, Illinois, May, physics, *The determination of the Peirce and Scheffers algebras of order eight.*

Fay Farnum, Cornell, June, physics, *Triadic Cremona nets of plane curves.*

Orrin Frink, Columbia, December, *The operations of boolean algebras.*

R. J. Garver, Chicago, September, I: *On Tschirnhaus transformations*; II: *Division algebras of order sixteen.*

J. S. Georges, Chicago, September, *Associativity conditions for division algebras corresponding to any abelian group.*

R. F. Graesser, Illinois, May, statistics and physics, *A certain general type of Neumann expansions and expansions in confluent hypergeometric functions.*

Marian C. Gray, Bryn Mawr, June, physics, *Theory of singular ordinary differential equations of the second order.*

L. S. Hill, Yale, June, *Aggregate functions and an application in analysis situs.*

H. M. Hosford, Illinois, May, physics, *On the summability of Fourier-Bessel and Dini expansions.*

C. M. Huber, Illinois, May, theoretical physics, *On complete systems of irrational invariants of associated point sets.*

F. E. Johnston, Illinois, May, astronomy, *Transitive substitution groups containing a regular subgroup of lower degree.*

B. F. Kimball, Cornell, June, physics, *Geodesics on a toroid.*

F. W. Kokomoor, Michigan, June, physics, *The teaching of elementary geometry in the seventeenth century.*

B. O. Koopman, Harvard, *On rejection to infinity and exterior motion in the restricted problem of three bodies.*

B. C. Patterson, Johns Hopkins, June, geophysics, *The algebraic and differential invariants of inversive geometry.*

H. R. Phalen, Chicago, June, *Metric properties of the quadric of Moutard.*

I. R. Pounder, Chicago, September, *A method of successive approximations for a partial equation of hyperbolic type.*

H. H. Pryde, New York University, June, *On a certain quintic curve with a triple point.*

D. E. Richmond, Cornell, June, physics, *Geodesics on surfaces of genus zero with knobs.*

C. F. Roos, Rice Institute, June, theoretical economics, *Generalized Lagrange problems in the calculus of variations.*

P. D. Schwartz, Yale, June, *Studies in non-euclidean geometry.*

M. M. Slotnick, Harvard, *Fundamental transformations of surfaces.*

H. L. Smith, Chicago, September, *The Minkowski linear measure for a simple rectifiable curve.*

P. A. Smith, Princeton, *Approximation of curves and surfaces by algebraic curves and surfaces.*

Marion E. Stark, Chicago, September, *A self-adjoint boundary value problem associated with a problem of the calculus of variations.*

Guy Stevenson, Illinois, May, physics, *Expansions of the Neumann type in terms of products of Bessel functions.*

M. H. Stone, Harvard, *Ordinary linear homogeneous differential equations of order n and the related expansion problems.*

Takashi Terami, California, May, physics, *The solution of the differential equation of a vibrating membrane by successive approximations.*

E. T. Virata, Johns Hopkins, June, geophysics, *W-surfaces which have an isometric spherical representation of their lines of curvature.*

P. S. Wagner, Johns Hopkins, June, geophysics, *An extension of Clifford's chain.*

The following graduate courses in mathematics are announced for the summer of 1927:

BROWN UNIVERSITY, June 20 to July 30.—By Professor R. E. Langer: Theory of integral equations.—By Professor M. H. Ingraham: Finite groups and the Galois theory of equations.

UNIVERSITY OF CHICAGO, first term, June 20 to July 27; second term, July 27 to September 2.—By Professor G. A. Bliss: Calculus of variations;

Thesis work in analysis.—By Professor H. E. Slaught: Differential equations; Theory of definite integrals.—By Professor E. T. Bell: Theory of functions of a complex variable; Theory of modular systems; Reading and research in the theory of numbers.—By Professor H. H. Mitchell: Introduction to higher algebra; Analytic theory of numbers; Reading and research in algebra and the theory of numbers.—By Professor W. C. Graustein: Analytic projective geometry; Metric differential geometry; Reading and research in differential geometry.—By Professor Mayme I. Logsdon: Algebraic geometry of hyper-spaces; Reading and research in algebraic geometry.—By Professor Warren Weaver: Electro-dynamics; Reading and research in electro-dynamics.—By Professor Walter Bartky: Celestial mechanics; descriptive astronomy.

COLUMBIA UNIVERSITY, July 11 to August 19.—By Professor E. R. Hedrick: Theory of functions of a real variable; Fundamental concepts of mathematics.—By Professor W. B. Fite: Higher algebra.—By Professor G. A. Pfeiffer: Projective geometry.—By Professor K. W. Lamson: Differential equations.

UNIVERSITY OF ILLINOIS, June 20 to August 13.—By Professor J. B. Shaw: Synoptic course in higher mathematics.—By Professor R. D. Carmichael: Relativity.—By Professor Arnold Emch: Geometric transformations; Mathematics of statistics.— By Dr. M. G. Carman: Advanced calculus.—By Dr. F. E. Johnston: Theory of equations.

UNIVERSITY OF IOWA, first term, June 13 to July 22; second term, July 25 to August 26.—First term: Dr. M. A. Nordgaard: Subject matter and teaching of mathematics.—By Dr. L. E. Ward: Ordinary differential equations; Differential geometry.—By Professor C. C. Wylie: Astronomy; Mathematics of finance; Celestial mechanics.—By Professor Roscoe Woods: Projective geometry.—Second term: By Dr. M. A. Nordgaard; History of mathematics.—By Dr. N. B. Conkwright: Differential equations.—By Professor E. W. Chittenden: Matrices and determinants; Introduction to the analysis of continua.

JOHNS HOPKINS UNIVERSITY, June 28 to August 5.—By Professor J. R. Musselman: Projective geometry.

UNIVERSITY OF MICHIGAN, June 27 to August 19.—By Professor W. B. Ford: Advanced calculus; Infinite series with special reference to Fourier series.—By Professor L. C. Karpinski: Teaching of geometry; History of mathematics.—By Professor Peter Field: Vector analysis.—By Professor T. R. Running: Empirical formulas.—By Professor T. H. Hildebrandt: Theory of functions of a complex variable.—By Professor H. C. Carver: Theory of probability; Advanced mathematical theory of statistics.—By Professor L. A. Hopkins: Elements of mechanics.—By Professor V. C. Poor: Differential equations.—By Professor C. J. Coe: Analytic mechanics; Integral equations.—By Professor Norman Anning: Solid analytic geometry; Finite differences.—By Professor J. A. Nys-

wander: Higher algebra.—By Professor G. Y. Rainich: Quadratic forms and quadratic numbers.—By Professor R. L. Wilder: Differential equations; Analysis situs.—By Mr. S. E. Field: Projective geometry.

UNIVERSITY OF MINNESOTA, first term, June 17 to July 29; second term, July 30 to September 3.—First term: By Professor DunhamJackson: History of ancient and modern mathematics.—By Professor W. L. Hart: Differential equations.—By Professors Hart and Jackson: Reading in advanced mathematics.—Second term: Professor R. W. Brink: Selected topics in advanced mathematics.

OHIO STATE UNIVERSITY, first term, June 18 to July 23; second term, July 25 to August 31.—By Professor Henry Blumberg: Introduction to the theory of functions of a complex variable; Problems in analysis; Reading and research in analysis.—By Professor H. W. Kuhn: Analytic projective geometry; Finite groups; Reading and research in group theory.

UNIVERSITY OF PENNSYLVANIA, July 5 to August 13.—By Professor M. J. Babb: Theory of numbers.—By Professor J. R. Kline: Theory of functions of a real variable.

STANFORD UNIVERSITY.—By Professor W. A. Manning: Modern algebra; Groups of finite order.—By Dr. Harold Hotelling: Analysis situs; Probability and statistics.—By Dr. H. W. Brinkmann: Differential equations; Theory of functions of a real variable.—By Miss Weiss: Theory of functions of a complex variable.

UNIVERSITY OF TEXAS, first term, June 8 to July 20; second term, July 20 to August 31.—First term: By Dean H. Y. Benedict: Advanced calculus.—By Professor E. L. Dodd: Functions of real variables; Probability.—By Professor R. L. Moore: Theory of sets; Foundations of geometry.—By Professor H. J. Ettlinger: Calculus of variations; Mathematical physics.—By Professor C. D. Rice: Advanced calculus.—By Professor P. M. Batchelder: Theory of equations.—Second term: By Professor A. C. Lunn: Relativity; Crystallography.—Professor H. J. Ettlinger: Calculus of variations; Mathematical physics.—By Professor H. S. Vandiver: Finite groups; Definite integrals.—By Professor Goldie Horton: Advanced calculus.

UNIVERSITY OF WISCONSIN, general session, June 25 to August 5.— By Professor L. W. Dowling: Projective geometry; Analytic geometry.— By Professor J. H. Taylor: Differential equations; Theory of equations.— By Professor R. W. Babcock: Analytical mechanics; Vector analysis.— Special nine weeks course for graduates, June 25 to August 26: Professor E. B. Skinner: Linear algebras; Theory of finite groups.

The Massachusetts Institute of Technology has received a grant of $230,000 from the Guggenheim Fund for the Promotion of Aeronautics; the gift will provide a building and equipment.

The University of Pittsburgh has appointed a number of engineers of the Westinghouse Company as part-time lecturers at the University in electrical and mechanical engineering, physics, and engineering mathematics; among these lecturers is Dr. Joseph Slepian.

Professor S. D. Wiksell, of Lund University, has been invited to lecture at the University of Michigan, during the coming academic year, on mathematical statistics.

Mr. S. F. Bibb, of the University of North Dakota, has been appointed assistant professor of mathematics at the Armour Institute of Technology.

Professor A. B. Coble of the University of Illinois has been appointed to a full professorship of mathematics in The Johns Hopkins University, and will assume his new duties in September, 1927.

Professor Arnold Dresden of the University of Wisconsin has been appointed to a full professorship of mathematics at Swarthmore College, and will assume his new duties in September, 1927.

Associate Professor C. B. Upton, of Teachers College, Columbia University, has been promoted to a full professorship of mathematics.

Haton de la Goupillière, formerly director of the Ecole Supérieure des Mines, and member of the Paris Academy of Sciences, has died, at the age of ninety-four.

Professor Carl Runge, of the University of Göttingen, died January 3, 1927, at the age of seventy-one.

Sir George Greenhill died February 17, 1927. He had been a member of the American Mathematical Society since 1897.

Mr. L. A. Anderson, formerly actuary of the Central Life Assurance Society, Des Moines, died January 20, 1927; he was a member of this Society.

Professor S. R. Cruse, of the University of Arizona, died February 8, 1926; he was a member of this Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BACHMANN (P.). Allgemeine Arithmetik der Zahlen-Körper. (Zahlentheorie, Band V.) Leipzig, Teubner, 1926. 548+22 pp.

BAEUMLER (A.) und SCHROETER (M.). Handbuch der Philosophie. 6te und 7te Lieferung: Logik und Systematik der Geisteswissenschaften, von E. Rothacker. München und Berlin, Oldenbourg, 1926.

BIERBERBACH (L.). See ENRIQUES (F.).

BOREL (E.). Space and time. London, Blackie, 1926. 14+234 pp.

BOUTROUX (P.). Das Wissenschaftsideal der Mathematiker. Uebersetzt von H. Pollaczek. (Wissenschaft und Hypothese, Band 28.) Leipzig, Teubner, 1927.

BREUSCH (F.). Was ein mathematisches Gymnasium leisten könnte. Karsruhe, G. Braun, 1926.

CHINI (M.). Esercizi di calcolo infinitesimale. 4a edizione. Livorno, Giusti, 1926. 320 pp.

CREAMGA (S. L.). Directions cyclifiablement conjuguées. Courbes à courbure normale constante. Jassy, Goldner, 1926.

DAKIN (A.). Practical mathematics. Part 2. London, Bell, 1926. 8+257+10 pp.

DULL (R. W.). Mathematics for engineers. New York, McGraw-Hill, 1926. 17+780 pp.

DURELL (C. V.) and WRIGHT (R. M.). An introduction to the calculus. London, Bell, 1926.

ENRIQUES (F.). Zur Geschichte der Logik. Grundlagen und Aufbau der Wissenschaft im Urteil der mathematischen Denker. Deutsch von L. Bieberbach. (Wissenschaft und Hypothese, Band 26.) Leipzig, Teubner, 1927.

EULER (L.). Opera omnia. Series I, volumen XV: Commentationes analyticae ad theoriam serierum infinitarum pertinentes, volumen secundum. Edit G. Faber. Leipzig, Teubner, 1927. 10+722 pp.

FABER (G.). See EULER (L.).

FRAENKEL (A.). Zehn Vorlesungen über die Grundlegung der Mengenlehre. (Wissenschaft und Hypothese, Band 31.) Leipzig, Teubner, 1927.

GRANT (J. D). Notes on the calculus of finite differences. Ann Arbor, Edwards, 1926.

HAERING (T. L.). Ueber Individualität in Natur- und Geisteswelt. Leipzig, Teubner, 1926. 6+114 pp.

HAUSDORFF (F.). Mengenlehre. 2te, neugearbeitete Auflage. Berlin, de Gruyter, 1927. 285 pp.

HEINRICH (M.). Logarithmentafeln. 2te, umgearbeitete Auflage. Berlin, Grotesche Verlagsbuchhandlung, 1926.

HENNIG (P.). Meine Lösung des Fermat-Problems. Berlin, Selbstverlag, 1926.

JALLER (A.). Doppeldenken. Grundlagen einer neuen Weltanschauung. Berlin, Schwetschke, 1926.

JOACHIMI (O.). Mathematische Formelsammlung. Halle, Buchhandlung des Waisenhauses, 1925.

LEHEUX (J. W. N.). Beginselen der nomographie. Deventer, Kluwer, 1926. 63 pp.

LEWIS (G. N.). The anatomy of science. New Haven, Yale University Press, 1926. 10+221 pp.

MEWES (R). Lösung der überviergradigen Gleichungen nebst Anwendung in der Technik. Berlin, Weyland, 1925. 48 pp.

OPPENHEIM (P.). Die natürliche Ordnung der Wissenschaften. Grundgesetze der vergleichenden Wissenschaftslehre. Jena, Fischer, 1926.

PAPELIER (G.). Exercices de géométrie moderne. 8 volumes. Paris, Vuibert, 1925–26. 132+84+88+128+104+136+128+96 pp.

PATERNO (E.). La Società Italiana delle Scienze detta dei XL. Il suo passato ed il suo avvenire. Roma, Bardi, 1926.

POLLACZEK (H.). See BOUTROUX (P.).

ROTHACKER (E.). See BAEUMLER (A.).

SALPETER (J.). Einführung in die höhere Mathematik für Naturforscher und Aerzte. 3te Auflage Jena, Fischer, 1926. 11+383 pp.

SCHMIDT (H.). Die Kant-Laplacesche Theorie. Ideen zur Weltanschauung von Immanuel Kant und Pierre Laplace. Leipzig, Kröner, 1925. 20+228 pp.

SCHROETER (M.). See BAEUMLER (A.).

SCHUH (F.). Het getalbegrip, in het bijzonder het onmeetbare getal. Groningen, Noordhoff, 1927.

SINGER (C.). See YELDHAM (F. A.).

STEFFENSEN (J. F.). Interpolation. Baltimore, Williams and Wilkins, 1927. 10+248 pp.

STRUIK (R.). See DE VRIES (H.).

DE VRIES (H.). Die vierte Dimension. Nach der zweiten holländischen Ausgabe ins Deutsche übertragen von R. Struik. Leipzig, Teubner, 1926. 10+167 pp.

WIJDENES (P.). Theorie der rekenkunde. Groningen, Noordhoff, 1926. 163 pp.

WRIGHT (R. M.). See DURELL (C. V.).

YELDHAM (F. A.). The story of reckoning in the middle ages. With an introduction by Charles Singer. London, Harrap, 1926. 96 pp.

PART II. APPLIED MATHEMATICS

ANDREWS (E. S.). The strength of materials. 2d edition, revised. London, Chapman and Hall, 1925. 618 pp.

AUERBACH (F). Physik in graphischen Darstellungen. 2te Auflage. Leipzig, Teubner, 1925. 12+286 pp.

BAUDOUIN (M.). La préhistoire par les étoiles. Paris, Maloine, 1926. 40+329 pp.

BECQUEREL (J.). L'art musical dans ses rapports avec la physique. (Extrait du Tome II du Cours de physique.) Paris, Hermann, 1926. 41 pp.

BENRATH (A.). Physikalische Chemie. Teil 2: Thermische und photochemische Gleichgewichts- und Geschwindigkeitslehre. Dresden, Steinkopff, 1925. 192 pp.

BISCONCINI (G.). Elementi di matematica finanziaria e attuariale. 3a edizione. Roma, Signorelli, 1927.

BLEICH (F.) und MELAN (E.). Die gewöhnlichen und partiellen Differenzengleichungen der Baustatik. Berlin, Springer, 1927, 8+350 pp.

BLIGH (N. M.). The evolution and development of the quantum theory. With a foreword by Max Planck. New York, Longmans, 1926. 112 pp.

BOUASSE (H.). Acoustique générale. Ondes aérienns. Paris, Librairie Delagrave, 1926. 24+544 pp.

BOULANGER (A.). Dynamique des solides tournantes. Phénomènes gyroscopiques. Théorie élémentaire et applications. Notes par T. Got et préface de G. Koenigs. Paris, Gauthier-Villars, 1926. 180 pp.

BOWLEY (A. L.). Elements of statistics. 5th edition, revised. London, P. S. King, 1926. 475 pp.

TYCHONIS BRAHE opera omnia. Tomus 13. Edidit I. L. E. Dreyer. Hauniae, Libraria Gyldendaliana, 1926. 4+398 pp.

DE BROGLIE (M.). X-rays. Translated by J. R. Clarke. London, Methuen, 1925. 13+204 pp.

BROWNE (A. D.). Tables and charts for the use of engineers and others. Cambridge, Heffer, 1925. 6+197 pp.

BRYANT (J. M.) and CORRELL (J. A.). Alternating current circuits. New York, McGraw-Hill, 1925.

BUREAU HYDROGRAPHIQUE INTERNATIONAL. Tables pour le calcul des marées par les constantes harmoniques. Cannes, Robaudi, 1926.

CARSON (J. R.). Electric circuit theory and the operational calculus. New York, McGraw-Hill, 1926. 9+197 pp.

CHENEVEAU (C.). See FÉRY (C.).

CHWOLSON (O. D.). Traité de physique. Tome supplémentaire: La physique de 1914 à 1926. 1re partie. Traduit du russe par A. Corvisy. Paris, Hermann, 1927. 339 pp.

CLARKE (J. R.). See DE BROGLIE (M.).

CORRELL (J. A.). See BRYANT (J. M.).

COSSERAT (F.). See GIBBS (J. W.).

CORVISY (A.). See CHWOLSON (O. D.).

CRANZ (C.). Lehrbuch der Ballistik. Band 2: Innere Ballistik; die Bewegung des Geschosses durch das Rohr und ihre Begleiterscheinungen. Herausgegeben von C. Cranz unter Mitwirkung von O. Poppenberg und O. von Eberhard. Berlin, Springer, 1926. 464 pp.

CRUM (W. L.) und PATTON (A. C.). Economic statistics. Chicago, A. W. Shaw, 1925. 12+493 pp.

DREYER (I. L. E.). See TYCHO BRAHE.

VON EBERHARD (O.). See CRANZ (C.).

ECKHART (L.). Konstruktive Abbildungsverfahren. Eine Einfürung in die neuren Methoden der darstellenden Geometrie. Wien, Springer, 1926. 4+119 pp.

EVERSHEIM (P.). Wellenlängenmessungen des Lichtes im sichtbaren und unsichtbaren Spektralbereich. Braunschweig, Vieweg, 1926. 5+111 pp.

FÉRY (C.), CHENEVEAU (C.), et PAILLARD (G.). Piles primaires et accumulateurs. Paris, Baillière, 1925. 684 pp.

FRANKLIN (W. S.) and TERMAN (F. E.). Transmission line theory and certain related topics. London, Constable, 1926. 320 pp.

FRESNEL (A.J.). Abhandlungen über die Beugung des Lichts. (Ostwalds Klassiker der extakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1925.

FRIEDEL (G.). Leçons de cristallographie professées à la Faculté des Sciences de Strasbourg. Paris, Berger-Levrault, 1926.

GABEAUD (M.). Introduction mathématique aux sciences techniques de l'ingénieur. 3e édition, revue et corrigée. Paris, Librairie de l'Enseignement Technique, 1926. 440 pp.

GENILLON (L.). Les groupes coordonnés d'équations physiques. Applications à quelques lois d'électricité et du magnétisme. Paris, Les Presses Universitaires de France, 1926. 32 pp.

GERLACH (W.). Materie, Elektrizität, Energie. Grundlagen und Ergebnisse der experimentellen Atomforschung. Dresden, Steinkopff, 1926. 11+291 pp.

GIBBS (J. W.). Principes élémentaires de mécanique statistique développés plus particulièrement en vue d'obtenir une base rationnelle de la thermodynamique. Traduction de F. Cosserat. Paris, Hermann, 1926. 196 pp.

GIBBS (R. W. M.). Building mathematics. London, Blackie, 1925. 11+192+11 pp.

GIUA (M.). Sviluppo storico dell'atomistica e della dottrina della valenza. Torino, Unione tipografico-editrice Torinese, 1926. 140 pp.

GOT (T.). See BOULANGER (A.).

GRAAFLAND (J. C. N.). See SCHUH (F.)

GUTENBERG (B.), herausgegeben von. Lehrbuch der Geophysik. Lieferung 1. Berlin, Gebrüder Borntraeger, 1926. 176 pp.

HEDGES (E. S.) and MYERS (J. E.). The problem of physico-chemical periodicity. London, Arnold, 1926. 95 pp.

HILLS (W. D.). Mechanics and applied mathematics. Part 2: Applied mathematics. London, University of London Press, 1926. 11+248 pp.

HINSHELWOOD (C. N.). Thermodynamics (for students of chemistry). New York, Dutton, 1926. 185 pp.

INSTITUT INTERNATIONAL DE CHIMIE SOLVAY. Structure et activité chimique. Paris, Gauthier-Villars, 1926. 14+672 pp.

JOLLEY (L. B. W.). Alternating current rectification and allied problems. A mathematical and practical treatment from the engineering

view-point. 2d edition, revised and enlarged. London, Chapman and Hall, 1926. 22+472 pp.

KENT (F. C.) and KENT (M. E.). Compound interest and annuity tables. New York, McGraw-Hill, 1926. 8+214 pp.

KENT (M. E.). See KENT (F. C.).

KOENIGS (G.). See BOULANGER (A.).

KOPACZEWSKI (W.). Les ions d'hydrogène. Signification, mesure, applications, données numériques. Paris, Gauthier-Villars, 1926. 9+322 pp.

KRAMER (W.). Einführung in die darstellende Geometrie. Teil 1: Senkrechte Projektion auf eine Tafel. Leipzig, Teubner, 1926.

LAEMEL (R.). Introduction to the theory of relativity. London, Simpkin, Marshall and Company, 1926. 66 pp.

MAYER (H.). La genèse des atomes. Les éléments constitutifs des atomes déterminés par étalons vibratoires gradués. Paris, Blanchard, 1925.

MELAN (E.) See BLEICH (F.).

MÜLLER (F. J.). Direkte (exakte) Lösung des einfachen Rückwärtseinschneidens im Raume Teil 1: Fundamentalaufgabe der Lufttopographie. Liebenwerda, R Reiss, 1925.

MYERS (J. E.). See HEDGES (E. S.).

PAILLARD (G.). See FÉRY (C.).

PARVULESCO (C.). Les amas globulaires d'étoiles et leurs relations dans l'espace. Paris, Gauthier-Villars, 1925. 142 pp.

PATTON (A. C.). See CRUM (W. L.).

PHAFF (—.). La recherche des constantes harmoniques, la prédiction de la marée et du courant, et leur description à l'aide de ces constantes. Monaco, Bureau Hydrographique International, 1926. 136 pp.

PLANCK (M.). See BLIGH (N. M.).

POPPENBERG (O.). See CRANZ (C.).

RESEARCH DEPARTMENT, WOOLWICH Description and discussion of the air disturbance, round bullets in free flight London, H. M. Stationery Office, 1925. 24 pp.+4 plates.

SCHUH (F.) en GRAAFLAND (J C.N.). Vraagstukken over theoretische mechanica. Eerste deel. Groningen, Noordhoff, 1926. 265 pp.

SNOW (A. J.). Matter and gravity in Newton's time. London, Oxford University Press, 1926. 256 pp.

SPREEN (W.). Die physikalischen Grundlagen der Radiotechnik. 3te verbesserte Auflage. (Bibliothek des Radio-Amateurs.) Berlin, Springer, 1925. 163 pp.

STONER (E. C.). Magnetism and atomic structure. London, Methuen, 1926. 13+371 pp.

TERMAN (F. E.). See FRANKLIN (W. S.).

VERONNET (A.). Constitution et évolution de l'univers. Paris, Doin, 1926. 480 pp.

WIRTH (W.). Die Zeitwahrnehmung. Leipzig, E. Pfeiffer, 1926.

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9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: l. Similarly, 0 means "zero"; to indicate "cap O", backspace and overprint period; thus: O .
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★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\varepsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\pi}\ \bar{\omega}\ \overline{\Gamma}$

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \dot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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Evanston Colloquium Lectures, by FELIX KLEIN. 1893. Second Edition, New York, 1911. Price $1.25; to members, $0.85.

Mathematical Papers of the Chicago Congress, 1893. New York, 1896. Price $2.75; to members, $2.25.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

VANCOUVER, BRITISH COLUMBIA, June 18, 1927. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, University of California, Berkeley, not later than May 27.

MADISON, WISCONSIN, SUMMER MEETING AND COLLOQUIUM. September 6-10, 1927.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than August 8. Series of colloquium lectures will be delivered by Professor E. T. Bell on *Algebraic arithmetic*, and by Professor Anna Pell-Wheeler on *The theory af quadratic forms in infinitely many variables, and applications.*

NEW YORK CITY, October 29, 1927.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than October 8. At the request of the Program Committee, Professor F. D. Murnaghan will give an address on *Modern hydrodynamical theory, with special reference to aeronautics.*

BERKELEY, CALIFORNIA, October 29, 1927. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, Berkeley, California, not later than October 8. At the request of the Program Committee, Professor E. R. Hedrick will give an address on *Extensions of the theory of functions of a complex variable.*

ST. LOUIS, MISSOURI, November 26, 1927. (The Southwestern Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor E. B. Stouffer, Lawrence, Kansas, not later than November 5.

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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Changes of address of members should be communicated at once to the Secretary of the Society, R. G. D. RICHARDSON, 501 West 116th Street, New York.

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